

Pseudodifferential operators of infinite order in spaces of tempered ultradistributions

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Abstract

Specific global symbol classes and corresponding pseudodifferential operators of infinite order that act continuously on the space of tempered ultradistributions of Beurling and Roumieu type are constructed. For these classes, symbolic calculus is developed.

Mathematics Subject Classification 47G30, 46F05

Keywords ultradistributions, pseudodifferential operators

0 Introduction

Pseudodifferential operators that act continuously on Gevrey classes were vastly studied during the years. A lot of local symbol classes that give rise to such operators (both of finite and infinite order) were constructed by many authors. Also, global symbol classes and corresponding operators (of finite and infinite order), as well as their symbolic calculus were developed in [1], [2], [3], [4], [5], [6] (see also [13]). The functional frame in which those were studied are the Gelfand - Shilov spaces of Roumieu type. The symbol classes developed there are well suited for studying polyhomogeneous operators. In this paper we develop a global calculus for some classes of pseudodifferential operators of infinite order. The functional frame in which the considered symbol classes and the corresponding pseudodifferential operators will be studied is going to be Komatsu ultradistributions, more precisely the spaces of tempered ultradistributions of Beurling and Roumieu type. Our symbol classes are similar to those in [3] and [4], but the weights that control the growth of the derivatives of the symbols are constructed in such way that they give well suited environment for studying Anti-Wick and Weyl operators on the space of tempered ultradistributions. In this paper, we develop calculus for our symbol classes, i.e. we prove results about change of quantization, composition of operators and asymptotic expansion of the symbol of the transposed operator.

The paper is organized as follows:

1. **Preliminaries.** Definition and basic facts are given concerning test spaces and corresponding spaces of ultradistributions. Some facts are cited from [16] which will be needed for the next sections. Also, a kernel theorem is proven for the space of tempered

ultradistributions.

2. Definition and basic properties of the symbol classes. The definition of the symbol classes is given as well as their basic topological properties. Pseudodifferential operators $\text{Op}_\tau(a)$, arising from τ - quantization of the symbol a are studied. A theorem that gives the hypocontinuity of the mapping $(a, u) \mapsto \text{Op}_\tau(a)u$, for u in the test space, is proven.

3. Symbolic calculus. The space of asymptotic expansion is defined. Results, concerning change of quantization, composition of operators and asymptotic expansion of the symbol of the transposed operator are proven.

1 Preliminaries

The sets of natural, integer, positive integer, real and complex numbers are denoted by \mathbb{N} , \mathbb{Z} , \mathbb{Z}_+ , \mathbb{R} , \mathbb{C} . We use the symbols for $x \in \mathbb{R}^d$: $\langle x \rangle = (1 + |x|^2)^{1/2}$, $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_d}$, $D_j^{\alpha_j} = i^{-1} \partial^{\alpha_j} / \partial x^{\alpha_j}$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}^d$. If $z \in \mathbb{C}^d$, by z^2 we will denote $z_1^2 + \dots + z_d^2$. Note that, if $x \in \mathbb{R}^d$, $x^2 = |x|^2$.

Following [9], we denote by M_p a sequence of positive numbers $M_0 = 1$ so that:

$$(M.1) \quad M_p^2 \leq M_{p-1} M_{p+1}, \quad p \in \mathbb{Z}_+;$$

$$(M.2) \quad M_p \leq c_0 H^p \min_{0 \leq q \leq p} \{M_{p-q} M_q\}, \quad p, q \in \mathbb{N}, \text{ for some } c_0, H \geq 1;$$

$$(M.3) \quad \sum_{p=q+1}^{\infty} \frac{M_{p-1}}{M_p} \leq c_0 q \frac{M_q}{M_{q+1}}, \quad q \in \mathbb{Z}_+,$$

although in some assertions we could assume the weaker ones $(M.2)'$ and $(M.3)'$ (see [9]). For a multi-index $\alpha \in \mathbb{N}^d$, M_α will mean $M_{|\alpha|}$, $|\alpha| = \alpha_1 + \dots + \alpha_d$. Recall, $m_p = M_p / M_{p-1}$, $p \in \mathbb{Z}_+$ and the associated function for the sequence M_p is defined by

$$M(\rho) = \sup_{p \in \mathbb{N}} \log_+ \frac{\rho^p}{M_p}, \quad \rho > 0.$$

It is non-negative, continuous, monotonically increasing function, which vanishes for sufficiently small $\rho > 0$ and increases more rapidly than $(\ln \rho)^p$ when ρ tends to infinity, for any $p \in \mathbb{N}$.

Let $U \subseteq \mathbb{R}^d$ be an open set and $K \subset\subset U$ (we will use always this notation for a compact subset of an open set). Then $\mathcal{E}^{\{M_p\}, h}(K)$ is the space of all $\varphi \in \mathcal{C}^\infty(U)$ which satisfy $\sup_{\alpha \in \mathbb{N}^d} \sup_{x \in K} \frac{|D^\alpha \varphi(x)|}{h^\alpha M_\alpha} < \infty$ and $\mathcal{D}_K^{\{M_p\}, h}$ is the space of all $\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ with supports

in K , which satisfy $\sup_{\alpha \in \mathbb{N}^d} \sup_{x \in K} \frac{|D^\alpha \varphi(x)|}{h^\alpha M_\alpha} < \infty$;

$$\mathcal{E}^{(M_p)}(U) = \varprojlim_{K \subset\subset U} \varprojlim_{h \rightarrow 0} \mathcal{E}^{\{M_p\}, h}(K), \quad \mathcal{E}^{\{M_p\}}(U) = \varprojlim_{K \subset\subset U} \varinjlim_{h \rightarrow \infty} \mathcal{E}^{\{M_p\}, h}(K),$$

$$\mathcal{D}^{(M_p)}(U) = \varprojlim_{K \subset\subset U} \varprojlim_{h \rightarrow 0} \mathcal{D}_K^{\{M_p\}, h}, \quad \mathcal{D}^{\{M_p\}}(U) = \varprojlim_{K \subset\subset U} \varinjlim_{h \rightarrow \infty} \mathcal{D}_K^{\{M_p\}, h}.$$

The spaces of ultradistributions and ultradistributions with compact support of Beurling and Roumieu type are defined as the strong duals of $\mathcal{D}^{(M_p)}(U)$ and $\mathcal{E}^{(M_p)}(U)$, resp. $\mathcal{D}^{\{M_p\}}(U)$ and $\mathcal{E}^{\{M_p\}}(U)$. For the properties of these spaces, we refer to [9], [10] and [11]. In the future we will not emphasize the set U when $U = \mathbb{R}^d$. Also, the common notation for the symbols (M_p) and $\{M_p\}$ will be $*$.

For $f \in L^1$, its Fourier transform is defined by $(\mathcal{F}f)(\xi) = \int_{\mathbb{R}^d} e^{-ix\xi} f(x) dx$, $\xi \in \mathbb{R}^d$.

By \mathfrak{R} is denoted a set of positive sequences which monotonically increases to infinity. For $(r_p) \in \mathfrak{R}$, consider the sequence $N_0 = 1$, $N_p = M_p \prod_{j=1}^p r_j$, $p \in \mathbb{Z}_+$. One easily sees that this sequence satisfies (M.1) and (M.3)' and its associated function will be denoted by $N_{r_p}(\rho)$, i.e. $N_{r_p}(\rho) = \sup_{p \in \mathbb{N}} \log_+ \frac{\rho^p}{M_p \prod_{j=1}^p r_j}$, $\rho > 0$. Note, for given (r_p) and every $k > 0$ there is $\rho_0 > 0$ such that $N_{r_p}(\rho) \leq M(k\rho)$, for $\rho > \rho_0$. In [16] the following lemmas are proven (for the definition of subordinate function see [9]).

Lemma 1.1. *let $g : [0, \infty) \rightarrow [0, \infty)$ be an increasing function that satisfies the following estimate: for every $L > 0$ there exists $C > 0$ such that $g(\rho) \leq M(L\rho) + \ln C$. Then there exists subordinate function $\epsilon(\rho)$ such that $g(\rho) \leq M(\epsilon(\rho)) + \ln C'$, for some constant $C' > 1$.*

Lemma 1.2. *Let $(k_p) \in \mathfrak{R}$. There exists $(k'_p) \in \mathfrak{R}$ such that $k'_p \leq k_p$ and*

$$\prod_{j=1}^{p+q} k'_j \leq 2^{p+q} \prod_{j=1}^p k'_j \cdot \prod_{j=1}^q k'_j, \text{ for all } p, q \in \mathbb{Z}_+.$$

Hence, for every $(k_p) \in \mathfrak{R}$, we can find $(k'_p) \in \mathfrak{R}$, as lemma 1.2, such that $N_{k_p}(\rho) \leq N_{k'_p}(\rho)$, $\rho > 0$ and the sequence $N_0 = 1$, $N_p = M_p \prod_{j=1}^p k'_j$, $p \in \mathbb{Z}_+$, satisfies (M.2) if M_p does.

From now on, we always assume that M_p satisfies (M.1), (M.2) and (M.3). It is said that $P(\xi) = \sum_{\alpha \in \mathbb{N}^d} c_\alpha \xi^\alpha$, $\xi \in \mathbb{R}^d$, is an ultrapolynomial of the class (M_p) , resp. $\{M_p\}$, whenever the coefficients c_α satisfy the estimate $|c_\alpha| \leq CL^{|\alpha|}/M_\alpha$, $\alpha \in \mathbb{N}^d$ for some $L > 0$ and $C > 0$, resp. for every $L > 0$ and some $C_L > 0$. The corresponding operator $P(D) = \sum_{\alpha} c_\alpha D^\alpha$ is an ultradifferential operator of the class (M_p) , resp. $\{M_p\}$ and they act continuously on $\mathcal{E}^{(M_p)}(U)$ and $\mathcal{D}^{(M_p)}(U)$, resp. $\mathcal{E}^{\{M_p\}}(U)$ and $\mathcal{D}^{\{M_p\}}(U)$ and the corresponding spaces of ultradistributions. In [16] a special class of ultrapolynomials of class $*$ were constructed. We summarize the results obtained there in the following proposition.

Proposition 1.1. *Let $c > 0$ and $k > 0$, resp. $c > 0$ and $(k_p) \in \mathfrak{R}$ are arbitrary but fixed. Then there exist $l > 0$ and $q \in \mathbb{Z}_+$, resp. there exist $(l_p) \in \mathfrak{R}$ and $q \in \mathbb{Z}_+$ such that*

$$P_l(z) = \prod_{j=q}^{\infty} \left(1 + \frac{z^2}{l^2 m_j^2}\right), \text{ resp. } P_{l_p}(z) = \prod_{j=q}^{\infty} \left(1 + \frac{z^2}{l_j^2 m_j^2}\right), \text{ is an entire function that doesn't}$$

have zeroes on the strip $W = \mathbb{R}^d + i\{y \in \mathbb{R}^d \mid |y_j| \leq c, j = 1, \dots, d\}$. $P_l(x)$, resp. $P_{l_p}(x)$, is an ultrapolynomial of class $$. Moreover $|P_l(z)| \geq \tilde{C} e^{M(|z|/k)}$, resp. $|P_{l_p}(z)| \geq \tilde{C} e^{N_{k_p}(|z|)}$, $z \in$*

W , for some $\tilde{C} > 0$ and $\left| \partial_x^\alpha \frac{1}{P_l(x)} \right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} e^{-M(|x|/k)}$, resp. $\left| \partial_x^\alpha \frac{1}{P_{l_p}(x)} \right| \leq C \cdot \frac{\alpha!}{r^{|\alpha|}} e^{-N_{k_p}(|x|)}$,

$x \in \mathbb{R}^d$, $\alpha \in \mathbb{N}^d$, where C depends on k and l , resp. (k_p) and (l_p) , and M_p ; $r \leq c$ arbitrary but fixed.

We denote by $\mathcal{S}_2^{M_p, m}(\mathbb{R}^d)$, $m > 0$, the space of all smooth functions φ which satisfy

$$\sigma_{m,2}(\varphi) := \left(\sum_{\alpha, \beta \in \mathbb{N}^d} \int_{\mathbb{R}^d} \left| \frac{m^{|\alpha|+|\beta|} \langle x \rangle^{|\alpha|} D^\beta \varphi(x)}{M_\alpha M_\beta} \right|^2 dx \right)^{1/2} < \infty,$$

supplied with the topology induced by the norm $\sigma_{m,2}$. The spaces $\mathcal{S}'^{(M_p)}$ and $\mathcal{S}'^{\{M_p\}}$ of tempered ultradistributions of Beurling and Roumieu type respectively, are defined as the strong duals of the spaces $\mathcal{S}^{(M_p)} = \varprojlim_{m \rightarrow \infty} \mathcal{S}_2^{M_p, m}(\mathbb{R}^d)$ and $\mathcal{S}^{\{M_p\}} = \varinjlim_{m \rightarrow 0} \mathcal{S}_2^{M_p, m}(\mathbb{R}^d)$, respectively. In [7] (see also [14]) it is proved that the sequence of norms $\sigma_{m,2}$, $m > 0$, is equivalent with the sequences of norms $\|\cdot\|_m$, $m > 0$, where $\|\varphi\|_m := \sup_{\alpha \in \mathbb{N}^d} \frac{m^{|\alpha|} \|D^\alpha \varphi(\cdot) e^{M(|\cdot|)}\|_{L^\infty}}{M_\alpha}$. If

we denote by $\mathcal{S}_\infty^{M_p, m}(\mathbb{R}^d)$ the space of all infinitely differentiable functions on \mathbb{R}^d for which the norm $\|\cdot\|_m$ is finite (obviously it is a Banach space), then $\mathcal{S}^{(M_p)}(\mathbb{R}^d) = \varprojlim_{m \rightarrow \infty} \mathcal{S}_\infty^{M_p, m}(\mathbb{R}^d)$ and $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d) = \varinjlim_{m \rightarrow 0} \mathcal{S}_\infty^{M_p, m}(\mathbb{R}^d)$. Also, for $m_2 > m_1$, the inclusion $\mathcal{S}_\infty^{M_p, m_2}(\mathbb{R}^d) \rightarrow$

$\mathcal{S}_\infty^{M_p, m_1}(\mathbb{R}^d)$ is a compact mapping. So, $\mathcal{S}^*(\mathbb{R}^d)$ is a (FS) -space in (M_p) case, resp. a (DFS) -space in the $\{M_p\}$ case. Moreover, they are nuclear spaces. In [7] (see also [15]) it is proved that $\mathcal{S}^{\{M_p\}} = \varprojlim_{(r_i), (s_j) \in \mathfrak{R}} \mathcal{S}_{(r_p), (s_q)}^{M_p}$, where $\mathcal{S}_{(r_p), (s_q)}^{M_p} = \{\varphi \in \mathcal{C}^\infty(\mathbb{R}^d) \mid \|\varphi\|_{(r_p), (s_q)} < \infty\}$

and $\|\varphi\|_{(r_p), (s_q)} = \sup_{\alpha \in \mathbb{N}^d} \frac{\|D^\alpha \varphi(x) e^{N_{s_p}(|x|)}\|_{L^\infty}}{M_\alpha \prod_{p=1}^{|\alpha|} r_p}$. Also, the Fourier transform is a topological automorphism of \mathcal{S}^* and of \mathcal{S}'^* .

We need the following kernel theorem for \mathcal{S}'^* . The (M_p) case was already considered in [12] (the authors used the characterization of Fourier-Hermite coefficients of the elements of the space in the proof of the kernel theorem).

Proposition 1.2. *The following isomorphisms of locally convex spaces hold*

$$\begin{aligned} \mathcal{S}^*(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}^*(\mathbb{R}^{d_2}) &\cong \mathcal{S}^*(\mathbb{R}^{d_1+d_2}) \cong \mathcal{L}_b(\mathcal{S}'^*(\mathbb{R}^{d_1}), \mathcal{S}^*(\mathbb{R}^{d_2})), \\ \mathcal{S}'^*(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}'^*(\mathbb{R}^{d_2}) &\cong \mathcal{S}'^*(\mathbb{R}^{d_1+d_2}) \cong \mathcal{L}_b(\mathcal{S}^*(\mathbb{R}^{d_1}), \mathcal{S}'^*(\mathbb{R}^{d_2})). \end{aligned}$$

Proof. Note that $\mathcal{S}^*(\mathbb{R}^{d_1}) \otimes \mathcal{S}^*(\mathbb{R}^{d_2})$ is dense in $\mathcal{S}^*(\mathbb{R}^{d_1+d_2})$. This is true because of the continuous and dense inclusion $\mathcal{D}^*(\mathbb{R}^{d_1+d_2}) \rightarrow \mathcal{S}^*(\mathbb{R}^{d_1+d_2})$ and because $\mathcal{D}^*(\mathbb{R}^{d_1}) \otimes \mathcal{D}^*(\mathbb{R}^{d_2})$ is dense in $\mathcal{D}^*(\mathbb{R}^{d_1+d_2})$ (see theorem 2.1 of [10]). We need to prove that $\mathcal{S}^*(\mathbb{R}^{d_1+d_2})$ induces on $\mathcal{S}^*(\mathbb{R}^{d_1}) \otimes \mathcal{S}^*(\mathbb{R}^{d_2})$ the topology $\pi = \epsilon$ (the π and the ϵ topologies are the same because \mathcal{S}^* is nuclear). Because the bilinear mapping $(\varphi, \psi) \mapsto \varphi \otimes \psi$, $\mathcal{S}^*(\mathbb{R}^{d_1}) \times \mathcal{S}^*(\mathbb{R}^{d_1+d_2}) \rightarrow \mathcal{S}^*(\mathbb{R}^{d_1+d_2})$ is separately continuous it follows that it is continuous. This is true in the (M_p) case because $\mathcal{S}^{(M_p)}$ is (FS) -space (hence a F -space) and it is true in the

$\{M_p\}$ case because $\mathcal{S}^{\{M_p\}}$ is (DFS) - space (hence a barreled (DF) - space). The continuity of this bilinear mapping proves that the inclusion $\mathcal{S}^*(\mathbb{R}^{d_1}) \otimes_\pi \mathcal{S}^*(\mathbb{R}^{d_2}) \longrightarrow \mathcal{S}^*(\mathbb{R}^{d_1+d_2})$ is continuous, hence the topology π is stronger than the induced one. Let A' and B' be equicontinuous subsets of $\mathcal{S}^*(\mathbb{R}^{d_1})$ and $\mathcal{S}^*(\mathbb{R}^{d_2})$, respectively. There exist $h > 0$ and $C > 0$ such that $\sup_{T \in A'} |\langle T, \varphi \rangle| \leq C \|\varphi\|_h$ and $\sup_{F \in B'} |\langle F, \psi \rangle| \leq C \|\psi\|_h$ in the (M_p) case, resp. there exist $(k_p), (k'_p) \in \mathfrak{R}$ and $C > 0$ such that $\sup_{T \in A'} |\langle T, \varphi \rangle| \leq C \|\varphi\|_{(k_p), (k'_p)}$ and $\sup_{F \in B'} |\langle F, \psi \rangle| \leq C \|\psi\|_{(k_p), (k'_p)}$ in the $\{M_p\}$ case. We consider first the $\{M_p\}$ case. By lemma 1.2, without losing generality we can assume that $\prod_{j=1}^{p+q} k_j \leq 2^{p+q} \prod_{j=1}^p k_j \prod_{j=1}^q k_j$, $p \in \mathbb{Z}_+$ and the same for (k'_j) . Put $r_j = k_j/(2H)$ and $r'_j = k'_j/(2H)$, $j \in \mathbb{Z}_+$. For all $T \in A'$ and $F \in B'$, we have

$$\begin{aligned} |\langle T_x \otimes F_y, \chi(x, y) \rangle| &= |\langle F_y, \langle T_x, \chi(x, y) \rangle \rangle| \leq C \sup_{y, \beta} \frac{|\langle T_x, D_y^\beta \chi(x, y) \rangle| e^{N_{k'_p}(|y|)}}{M_\beta \prod_{j=0}^{|\beta|} k_j} \\ &\leq C^2 \sup_{x, y, \alpha, \beta} \frac{|D_x^\alpha D_y^\beta \chi(x, y)| e^{N_{k'_p}(|x|)} e^{N_{k'_p}(|y|)}}{M_\alpha M_\beta \prod_{j=0}^{|\alpha|} k_j \prod_{j=0}^{|\beta|} k_j} \\ &\leq c_0^2 C^2 \sup_{x, y, \alpha, \beta} \frac{|D_x^\alpha D_y^\beta \chi(x, y)| e^{N_{r'_j}(|(x, y)|)}}{M_{\alpha+\beta} \prod_{j=0}^{|\alpha|+|\beta|} r_j} = c_0^2 C^2 \|\chi\|_{(r_p), (r'_p)}, \end{aligned}$$

where, in the third inequality we used proposition 3.6 of [9] for $N_{k'_p}(\lambda)$. Similarly, in the (M_p) case one obtains $\sup_{T \in A', F \in B'} |\langle T_x \otimes F_y, \chi(x, y) \rangle| \leq c_0^2 C^2 \|\chi\|_{hH}$. Hence, the ϵ topology on $\mathcal{S}^*(\mathbb{R}^{d_1}) \otimes \mathcal{S}^*(\mathbb{R}^{d_2})$ is weaker than the induced one from $\mathcal{S}^*(\mathbb{R}^{d_1+d_2})$. This gives the isomorphism $\mathcal{S}^*(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}^*(\mathbb{R}^{d_2}) \cong \mathcal{S}^*(\mathbb{R}^{d_1+d_2})$. Proposition 50.5 of [20] yields the isomorphisms $\mathcal{S}^*(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}^*(\mathbb{R}^{d_2}) \cong \mathcal{L}_b(\mathcal{S}'^*(\mathbb{R}^{d_1}), \mathcal{S}^*(\mathbb{R}^{d_2}))$ and $\mathcal{S}'^*(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}'^*(\mathbb{R}^{d_2}) \cong \mathcal{L}_b(\mathcal{S}^*(\mathbb{R}^{d_1}), \mathcal{S}'^*(\mathbb{R}^{d_2}))$ (\mathcal{S}'^* is a Montel space). Now, because $\mathcal{S}^{(M_p)}$ is (F) - space, theorem 9.9 of [17] gives the isomorphism $\mathcal{S}'^{(M_p)}(\mathbb{R}^{d_1}) \hat{\otimes} \mathcal{S}'^{(M_p)}(\mathbb{R}^{d_2}) \cong \mathcal{S}'^{(M_p)}(\mathbb{R}^{d_1+d_2})$. In the $\{M_p\}$ case, $\mathcal{S}^{\{M_p\}}$ is (DFS) - space, i.e. the strong dual of the (FS) - space $\mathcal{S}'^{\{M_p\}}$, hence this theorem implies the same isomorphism in the $\{M_p\}$ case. \square

2 Definition and basic properties of the symbol classes

Let $a \in \mathcal{S}'^*(\mathbb{R}^{2d})$. For $\tau \in \mathbb{R}$, consider the ultradistribution

$$K_\tau(x, y) = \mathcal{F}_{\xi \rightarrow x-y}^{-1}(a)((1-\tau)x + \tau y, \xi) \in \mathcal{S}'^*(\mathbb{R}^{2d}). \quad (1)$$

Let $\text{Op}_\tau(a)$ be the operator from \mathcal{S}^* to \mathcal{S}'^* corresponding to the kernel $K_\tau(x, y)$, i.e.

$$\langle \text{Op}_\tau(a)u, v \rangle = \langle K_\tau, v \otimes u \rangle, \quad u, v \in \mathcal{S}^*(\mathbb{R}^d). \quad (2)$$

a will be called the τ -symbol of the pseudo-differential operator $\text{Op}_\tau(a)$. When $\tau = 0$, we will denote $\text{Op}_0(a)$ by $a(x, D)$. When $a \in \mathcal{S}^*(\mathbb{R}^{2d})$,

$$\text{Op}_\tau(a)u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} a((1-\tau)x + \tau y, \xi) u(y) dy d\xi, \quad (3)$$

where the integral is absolutely convergent.

Proposition 2.1. *The correspondence $a \mapsto K_\tau$ is an isomorphism of $\mathcal{S}^*(\mathbb{R}^{2d})$, of $\mathcal{S}'^*(\mathbb{R}^{2d})$ and of $L^2(\mathbb{R}^{2d})$. The inverse map is given by*

$$a(x, \xi) = \mathcal{F}_{y \rightarrow \xi} K_\tau(x + \tau y, x - (1 - \tau)y).$$

Proof. The partial Fourier transform and the composition with the change of variable $\Xi(x, y) = ((1 - \tau)x + \tau y, x - y)$ are isomorphisms of $\mathcal{S}^*(\mathbb{R}^{2d})$, of $\mathcal{S}'^*(\mathbb{R}^{2d})$ and of $L^2(\mathbb{R}^{2d})$. The last part is just an easy computation. \square

Operators with symbols in \mathcal{S}^* correspond to kernels in \mathcal{S}^* and by proposition 1.2, those extend to continuous operators from \mathcal{S}'^* to \mathcal{S}^* . We will call these $*$ -regularizing operators.

Now we will define the announced global symbol classes. Let A_p and B_p be sequences that satisfy (M.1), (M.3)' and $A_0 = 1$ and $B_0 = 1$. Moreover, let $A_p \subset M_p$ and $B_p \subset M_p$ i.e. there exist $c_0 > 0$ and $L > 0$ such that $A_p \leq c_0 L^p M_p$ and $B_p \leq c_0 L^p M_p$, for all $p \in \mathbb{N}$ (it is obvious that without losing generality we can assume that this c_0 is the same with c_0 from the conditions (M.2) and (M.3) for M_p). For $0 < \rho \leq 1$, define $\Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; h, m)$ as the space of all $a \in C^\infty(\mathbb{R}^{2d})$ for which the following norm is finite

$$\sup_{\alpha, \beta} \sup_{(x, \xi) \in \mathbb{R}^{2d}} \frac{|D_\xi^\alpha D_x^\beta a(x, \xi)| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta|} e^{-M(m|\xi|)} e^{-M(m|x|)}}{h^{|\alpha| + |\beta|} A_\alpha B_\beta}.$$

It is easily verified that it is a Banach space. Define

$$\begin{aligned} \Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; m) &= \varprojlim_{h \rightarrow 0} \Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; h, m), \quad \Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}) = \varinjlim_{m \rightarrow \infty} \Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; m), \\ \Gamma_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; h) &= \varprojlim_{m \rightarrow 0} \Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; h, m), \quad \Gamma_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}) = \varinjlim_{h \rightarrow \infty} \Gamma_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; h). \end{aligned}$$

Remark 2.1. $\Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; m)$ and $\Gamma_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; h)$ are (F) -spaces. Obviously, the inclusion mappings $\Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; m) \rightarrow \mathcal{S}'^{(M_p)}(\mathbb{R}^{2d})$ and $\Gamma_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; h) \rightarrow \mathcal{S}'^{\{M_p\}}(\mathbb{R}^{2d})$ are continuous, hence $\Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d})$ and $\Gamma_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d})$ are Hausdorff l.c.s. Moreover, as inductive limits of barreled and bornological l.c.s., they are barreled and bornological.

Remark 2.2. By proposition 7 of [8] it follows that every element of $\Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ is a multiplier for $\mathcal{S}'^*(\mathbb{R}^{2d})$.

Remark 2.3. Examples of nontrivial elements of $\Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ are given by every ultra-polynomial of class $*$.

Proposition 2.2. *For every $a \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ there exists a sequence χ_j , $j \in \mathbb{Z}_+$, in $\mathcal{D}^*(\mathbb{R}^{2d})$ such that $\chi_j \rightarrow a$ in $\Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$.*

Proof. Let $\varphi(x) \in \mathcal{D}^{(B_p)}(\mathbb{R}^d)$ and $\psi(\xi) \in \mathcal{D}^{(A_p)}(\mathbb{R}^d)$, in the (M_p) case, resp. $\varphi(x) \in \mathcal{D}^{(B_p)}(\mathbb{R}^d)$ and $\psi(\xi) \in \mathcal{D}^{(A_p)}(\mathbb{R}^d)$ in the $\{M_p\}$ case, are such that $0 \leq \varphi, \psi \leq 1$, $\varphi(x) = 1$ when $|x| \leq 1/4$, $\psi(\xi) = 1$ when $|\xi| \leq 1/4$ and $\varphi(x) = 0$ when $|x| \geq 1/2$, $\psi(\xi) = 0$ when $|\xi| \geq 1/2$ (such functions exist because A_p and B_p satisfy $(M.3)'$). Put $\chi(x, \xi) = \varphi(x)\psi(\xi)$, $\chi_n(x, \xi) = \chi(x/n, \xi/n)$ for $n \in \mathbb{Z}_+$. Then $\chi, \chi_n \in \mathcal{D}^{(M_p)}(\mathbb{R}^{2d})$, resp. $\chi, \chi_n \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^{2d})$. For $a \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$, it is readily seen that $a_n(x, \xi) = \chi_n(x, \xi)a(x, \xi)$ is an element of $\mathcal{D}^*(\mathbb{R}^{2d})$. It is easy to prove that there exists $m > 0$ such that for every $h > 0$, $a_n \rightarrow a$ in $\Gamma_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; h, m)$ in the (M_p) case, resp. there exists $h > 0$ such that for every $m > 0$, $a_n \rightarrow a$ in $\Gamma_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; h, m)$ in the $\{M_p\}$ case. \square

Theorem 2.1. *Let $a \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$. Then the integral (3) is well defined as an iterated integral. The ultradistribution $\text{Op}_\tau(a)u$, $u \in \mathcal{S}^*$, coincides with the function defined by that iterated integral.*

Proof. The (M_p) case. Because $a \in \Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d})$, there exists $m > 0$ such that, for every $h > 0$ there exists $C_1 > 0$ such that

$$|D_\xi^\alpha D_x^\beta a(x, \xi)| \leq C_1 \frac{h^{|\alpha|+|\beta|} A_\alpha B_\beta e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x, \xi) \rangle^{\rho|\alpha|+\rho|\beta|}}, \quad \forall \alpha, \beta \in \mathbb{N}^d, \quad \forall (x, \xi) \in \mathbb{R}^{2d}.$$

Hence, $b(x, \xi) = \int_{\mathbb{R}^d} e^{i(x-y)\xi} a((1-\tau)x + \tau y, \xi) u(y) dy$ is well defined and $b \in \mathcal{C}^\infty(\mathbb{R}^{2d})$.

Choose $m_0 > 0$ large enough such that, for all $m' \geq m_0$, $\int_{\mathbb{R}^d} e^{M(2m|\tau y|)} e^{-M(m'|y|)} dy < \infty$.

Because $u \in \mathcal{S}^{(M_p)}$, for such m' we get $\sup_{\alpha \in \mathbb{N}^d} \frac{m'^{|\alpha|} \|D^\alpha u(y) e^{M(m'|y|)}\|_{L^\infty}}{M_\alpha} < \infty$. One obtains

$$\begin{aligned} & |\xi^\alpha b(x, \xi)| \\ &= \left| \int_{\mathbb{R}^d} e^{i(x-y)\xi} D_y^\alpha (a((1-\tau)x + \tau y, \xi) u(y)) dy \right| \\ &\leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \int_{\mathbb{R}^d} |\tau|^{|\gamma|} |D_x^\gamma a((1-\tau)x + \tau y, \xi)| |D^{\alpha-\gamma} u(y)| dy \\ &\leq C \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \int_{\mathbb{R}^d} |\tau|^{|\gamma|} h^{|\gamma|} B_\gamma e^{M(m|\xi|)} e^{M(m|(1-\tau)x + \tau y|)} \frac{M_{\alpha-\gamma} e^{-M(m'|y|)}}{m'^{|\alpha|-|\gamma|}} dy \\ &\leq C' \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \int_{\mathbb{R}^d} (|\tau| h L)^{|\gamma|} M_\gamma e^{M(m|\xi|)} e^{M(2m|(1-\tau)x|)} e^{M(2m|\tau y|)} \frac{M_{\alpha-\gamma} e^{-M(m'|y|)}}{m'^{|\alpha|-|\gamma|}} dy \\ &\leq C'' e^{M(m|\xi|)} e^{M(2m|(1-\tau)x|)} M_\alpha \left(|\tau| h L + \frac{1}{m'} \right)^{|\alpha|}, \end{aligned}$$

where we used $B_p \subset M_p$. For $l > 0$ consider $P_l(\xi)$. By proposition 1.1, we can choose l such that $|P_l(\xi)| \geq c'' e^{M(r|\xi|)}$ where $r > 0$ is such that $\int_{\mathbb{R}^d} e^{M(m|\xi|)} e^{-M(r|\xi|)} d\xi < \infty$ and $P_l(\xi)$ is never zero. Also, if we represent $P_l(\xi) = \sum_{\alpha} c_{\alpha} \xi^{\alpha}$, there exists $L' > 0$ and $C' > 0$ such that $|c_{\alpha}| \leq C' L'^{|\alpha|} / M_{\alpha}$. Choose $h > 0$ so small and $m' \geq m_0$ so large such that $\left(|\tau| h L + \frac{1}{m'}\right) L' < \frac{1}{4}$. Then, we have

$$\begin{aligned} |P_l(\xi) b(x, \xi)| &\leq \sum_{\alpha} |c_{\alpha}| |\xi^{\alpha} b(x, \xi)| \leq C'' e^{M(m|\xi|)} e^{M(2m|(1-\tau)x|)} \sum_{\alpha} |c_{\alpha}| M_{\alpha} \left(|\tau| h L + \frac{1}{m'}\right)^{|\alpha|} \\ &\leq C_0 e^{M(m|\xi|)} e^{M(2m|(1-\tau)x|)}. \end{aligned}$$

Hence $\int_{\mathbb{R}^d} |b(x, \xi)| d\xi$ is finite for every x , i.e. (3) is well defined as iterated integral. From this estimate also follows that $b(x, \xi) v(x) \in L^1(\mathbb{R}^{2d})$, for any $v \in \mathcal{S}^{(M_p)}$.

Let us consider the $\{M_p\}$ case. Because $a \in \Gamma_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d})$, there exists $h > 0$ such that, for every $m > 0$ there exists $C_1 > 0$ such that

$$|D_{\xi}^{\alpha} D_x^{\beta} a(x, \xi)| \leq C_1 \frac{h^{|\alpha|+|\beta|} A_{\alpha} B_{\beta} e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x, \xi) \rangle^{\rho|\alpha|+\rho|\beta|}}, \quad \forall \alpha, \beta \in \mathbb{N}^d, \quad \forall (x, \xi) \in \mathbb{R}^{2d}.$$

Hence, $b(x, \xi) = \int_{\mathbb{R}^d} e^{i(x-y)\xi} a((1-\tau)x + \tau y, \xi) u(y) dy$ is well defined and $b \in \mathcal{C}^{\infty}(\mathbb{R}^{2d})$. Put

$$g(\lambda) = \sup_{|(x, \xi)| \leq \lambda} \sup_{\alpha, \beta} \ln_+ \frac{|D_{\xi}^{\alpha} D_x^{\beta} a(x, \xi)| \langle (x, \xi) \rangle^{\rho|\alpha|+\rho|\beta|}}{h^{|\alpha|+|\beta|} A_{\alpha} B_{\beta}}.$$

g is an increasing function and by proposition 3.6 of [9], it satisfies the condition of lemma 1.1. Hence, there exists subordinate function $\epsilon(\lambda)$ and a constant $C' > 1$ such that $g(\lambda) \leq M(\epsilon(\lambda)) + \ln C'$. We get that

$$|D_{\xi}^{\alpha} D_x^{\beta} a(x, \xi)| \leq C' \frac{h^{|\alpha|+|\beta|} A_{\alpha} B_{\beta} e^{M(\epsilon(|(x, \xi)|))}}{\langle (x, \xi) \rangle^{\rho|\alpha|+\rho|\beta|}}, \quad \forall \alpha, \beta \in \mathbb{N}^d, \quad \forall (x, \xi) \in \mathbb{R}^{2d}.$$

By lemma 3.12 of [9], there exist another sequence \tilde{N}_p , which satisfies (M.1), such that $\tilde{N}(\lambda) \geq M(\epsilon(\lambda))$ and $k'_p = \tilde{n}_p / m_p \rightarrow \infty$ when $p \rightarrow \infty$. There exist $(k''_p) \in \mathfrak{R}$ such that $k''_p \leq k'_p$, for $p \in \mathbb{Z}_+$. Then

$$e^{N_{k''_p}(\lambda)} = \sup_p \frac{\lambda^p}{M_p \prod_{j=1}^p k''_j} \geq \sup_p \frac{\lambda^p}{M_p \prod_{j=1}^p k'_j} = e^{\tilde{N}(\lambda)} \geq e^{M(\epsilon(\lambda))}.$$

From this, we obtain the estimate

$$|D_{\xi}^{\alpha} D_x^{\beta} a(x, \xi)| \leq C \frac{h^{|\alpha|+|\beta|} A_{\alpha} B_{\beta} e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|x|)}}{\langle (x, \xi) \rangle^{\rho|\alpha|+\rho|\beta|}}, \quad \forall \alpha, \beta \in \mathbb{N}^d, \quad \forall (x, \xi) \in \mathbb{R}^{2d}, \quad (4)$$

where we choose $(k_p) \in \mathfrak{R}$ such that $e^{N_{k_p}''(|(x,\xi)|)} \leq c' e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|x|)}$, for some $c' > 0$. Because $u \in \mathcal{S}^{\{M_p\}}$, there exists $h_1 > 0$ such that for every $(s_p) \in \mathfrak{R}$, $\sup_{\alpha} \frac{h_1^{|\alpha|} \|e^{N_{s_p}(|x|)} D^{\alpha} u(x)\|_{L^{\infty}}}{M_{\alpha}} < \infty$. Choose $(s_p) \in \mathfrak{R}$, such that $\int_{\mathbb{R}^d} e^{N_{k_p/2}(|\tau y|)} e^{-N_{s_p}(|y|)} dy < \infty$. Then, we have

$$\begin{aligned}
& |\xi^{\alpha} b(x, \xi)| \\
&= \left| \int_{\mathbb{R}^d} e^{i(x-y)\xi} D_y^{\alpha} (a((1-\tau)x + \tau y, \xi) u(y)) dy \right| \\
&\leq \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \int_{\mathbb{R}^d} |\tau|^{|\gamma|} |D_x^{\gamma} a((1-\tau)x + \tau y, \xi)| |D^{\alpha-\gamma} u(y)| dy \\
&\leq C' \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \int_{\mathbb{R}^d} |\tau|^{|\gamma|} h^{|\gamma|} B_{\gamma} e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|(1-\tau)x + \tau y|)} \frac{e^{-N_{s_p}(|y|)} M_{\alpha-\gamma}}{h_1^{|\alpha|-|\gamma|}} dy \\
&\leq C'' \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \int_{\mathbb{R}^d} \frac{(|\tau| h L)^{|\gamma|} M_{\gamma} e^{N_{k_p}(|\xi|)} e^{N_{k_p/2}(|(1-\tau)x|)} e^{N_{k_p/2}(|\tau y|)} e^{-N_{s_p}(|y|)} M_{\alpha-\gamma}}{h_1^{|\alpha|-|\gamma|}} dy \\
&\leq C \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} \frac{(|\tau| h L)^{|\gamma|} e^{N_{k_p}(|\xi|)} e^{N_{k_p/2}(|(1-\tau)x|)} M_{\alpha}}{h_1^{|\alpha|-|\gamma|}} \\
&= C e^{N_{k_p}(|\xi|)} e^{N_{k_p/2}(|(1-\tau)x|)} M_{\alpha} \left(|\tau| h L + \frac{1}{h_1} \right)^{|\alpha|},
\end{aligned}$$

where we used $B_p \subset M_p$. For $(l_p) \in \mathfrak{R}$ consider $P_{l_p}(\xi)$. By proposition 1.1 we can choose $(l_p) \in \mathfrak{R}$ such that $|P_{l_p}(\xi)| \geq c'' e^{N_{r_p}(|\xi|)}$ where $(r_p) \in \mathfrak{R}$ is such that $\int_{\mathbb{R}^d} e^{N_{k_p}(|\xi|)} e^{-N_{r_p}(|\xi|)} d\xi < \infty$ and $P_{l_p}(\xi)$ is never zero. Also, if we represent $P_{l_p}(\xi) = \sum_{\alpha} c_{\alpha} \xi^{\alpha}$, then for any $L' > 0$ there exists $C' > 0$ such that $|c_{\alpha}| \leq C' L'^{|\alpha|} / M_{\alpha}$. Choose $L' > 0$ such that, $\left(|\tau| h L + \frac{1}{h_1} \right) L' < \frac{1}{4}$. By the above estimate, we have

$$\begin{aligned}
|P_{l_p}(\xi) b(x, \xi)| &\leq \sum_{\alpha} |c_{\alpha}| |\xi^{\alpha} b(x, \xi)| \leq C e^{N_{k_p}(|\xi|)} e^{N_{k_p/2}(|(1-\tau)x|)} \sum_{\alpha} |c_{\alpha}| M_{\alpha} \left(|\tau| h L + \frac{1}{h_1} \right)^{|\alpha|} \\
&\leq C_0 e^{N_{k_p}(|\xi|)} e^{N_{k_p/2}(|(1-\tau)x|)}.
\end{aligned}$$

Hence $\int_{\mathbb{R}^d} |b(x, \xi)| d\xi$ is finite for every x , i.e. (3) is well defined as iterated integral. From this estimate also follows that $b(x, \xi) v(x) \in L^1(\mathbb{R}^{2d})$, for any $v \in \mathcal{S}^{\{M_p\}}$.

Hence, in both cases we get that $\int_{\mathbb{R}^d} |b(x, \xi)| d\xi$ is finite for every x , i.e. (3) is well defined as iterated integral, and $b(x, \xi) v(x) \in L^1(\mathbb{R}^{2d})$, for any $v \in \mathcal{S}^*$. We will temporary denote $F(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} b(x, \xi) d\xi$. From the above estimates it is obvious that $F \in \mathcal{S}'^*$.

By Fubini's theorem, we have

$$\langle F, v \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} a((1-\tau)x + \tau y, \xi) u(y) v(x) dy dx d\xi.$$

By the growth condition of a , it is obvious that the integral

$$\int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} a((1-\tau)x + \tau y, \xi) u(y) v(x) dy dx$$

converges, so if we put the change of variable $\Xi(x, y) = ((1-\tau)x + \tau y, x - y)$ in the last term of the above equality we obtain $\langle F, v \rangle = \langle a, \mathcal{F}_2^{-1}((v \otimes u) \circ \Xi^{-1}) \rangle = \langle \text{Op}_\tau(a)u, v \rangle$, which completes the proof of the theorem. \square

We will define more general classes of operators and symbols.

Definition 2.1. Denote by $\Pi_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{3d}; h, m)$ the Banach space of all $a \in C^\infty(\mathbb{R}^{3d})$ with the norm

$$\sup_{\alpha, \beta, \gamma \in \mathbb{N}^d} \sup_{(x, y, \xi) \in \mathbb{R}^{3d}} \frac{|D_\xi^\alpha D_x^\beta D_y^\gamma a(x, y, \xi)| \langle (x, y, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + \rho|\gamma|} e^{-M(m|\xi|)} e^{-M(m|x|)} e^{-M(m|y|)}}{h^{|\alpha| + |\beta| + |\gamma|} \langle x - y \rangle^{\rho|\alpha| + \rho|\beta| + \rho|\gamma|} A_\alpha B_{\beta + \gamma}}$$

We will denote this norm by $\|\cdot\|_{h, m, \Pi}$. Define

$$\begin{aligned} \Pi_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{3d}; m) &= \varprojlim_{h \rightarrow 0} \Pi_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{3d}; h, m), \quad \Pi_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{3d}) = \varliminf_{m \rightarrow \infty} \Pi_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{3d}; m), \\ \Pi_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{3d}; h) &= \varprojlim_{m \rightarrow 0} \Pi_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{3d}; h, m), \quad \Pi_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{3d}) = \varliminf_{h \rightarrow \infty} \Pi_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{3d}; h). \end{aligned}$$

$\Pi_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{3d}; m)$ and $\Pi_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{3d}; h)$ are (F) -spaces. Similarly as for $\Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$, one proves that $\Pi_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{3d})$ are barreled and bornological l.c.s.

One easily sees that, for $a \in \Pi_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{3d})$, the function $b(x, \xi) = a(x, x, \xi)$ belongs to $\Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$. Moreover, if $p \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ and $\tau \in \mathbb{R}$, then $a(x, y, \xi) = p((1-\tau)x + \tau y, \xi)$ belongs to $\Pi_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{3d})$.

Remark 2.4. The Γ and Π classes defined here are appropriate generalization (for symbols of infinite order) in ultradistributional setting of the corresponding classes in the setting of Schwartz distributions (see [18] for the theory of distributions and [19] for the corresponding Γ and Π symbol classes and calculus in the setting of Schwartz distributions).

Lemma 2.1. Let $h > 0$ be fixed. For every bounded set B in $\Pi_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{3d}; h)$, there exist $C > 0$ and $(k_p) \in \mathfrak{R}$ such that, for all $a \in B$,

$$\sup_{\alpha, \beta, \gamma \in \mathbb{N}^d} \sup_{(x, y, \xi) \in \mathbb{R}^{3d}} \frac{|D_\xi^\alpha D_x^\beta D_y^\gamma a(x, y, \xi)| \langle (x, y, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + \rho|\gamma|} e^{-N_{k_p}(|\xi|)} e^{-N_{k_p}(|x|)} e^{-N_{k_p}(|y|)}}{h^{|\alpha| + |\beta| + |\gamma|} \langle x - y \rangle^{\rho|\alpha| + \rho|\beta| + \rho|\gamma|} A_\alpha B_{\beta + \gamma}} \leq C.$$

Proof. Because B is bounded, for every $m > 0$ there exists a constant $C_m > 0$ (which depends on m) such that, for every $a \in B$, $\|a\|_{h,m,\Pi} \leq C_m$, i.e.

$$\frac{|D_\xi^\alpha D_x^\beta D_y^\gamma a(x, y, \xi)| \langle (x, y, \xi) \rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}}{h^{|\alpha|+|\beta|+|\gamma|} \langle x-y \rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|} A_\alpha B_{\beta+\gamma}} \leq C_m e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m|y|)},$$

for all $(x, y, \xi) \in \mathbb{R}^{3d}$ and all $\alpha, \beta, \gamma \in \mathbb{N}^d$. Without losing generality, we can take $C_m \geq 1$. Put

$$g_a(x, y, \xi) = \sup_{\alpha, \beta, \gamma} \ln_+ \left(\frac{|D_\xi^\alpha D_x^\beta D_y^\gamma a(x, y, \xi)| \langle (x, y, \xi) \rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}}{h^{|\alpha|+|\beta|+|\gamma|} \langle x-y \rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|} A_\alpha B_{\beta+\gamma}} \right).$$

Then, by proposition 3.6 of [9], we have

$$\begin{aligned} g_a(x, y, \xi) &\leq M(m|\xi|) + M(m|x|) + M(m|y|) + \ln C_m \leq 3M(m|(x, y, \xi)|) + \ln C_m \\ &\leq M(mH^2|(x, y, \xi)|) + \ln(c_0^2 C_m). \end{aligned}$$

Now, define $\tilde{g}_a(\lambda) = \sup_{|(x, y, \xi)| \leq \lambda} g_a(x, y, \xi)$. Then $\tilde{g}_a(\lambda) \leq M(mH^2\lambda) + \ln(c_0^2 C_m)$, for $\lambda \geq 0$

and $a \in B$. Then, if we put $\tilde{g}(\lambda) = \sup_{a \in B} \tilde{g}_a(\lambda)$, we have $\tilde{g}(\lambda) \leq M(mH^2\lambda) + \ln(c_0^2 C_m)$, for

$\lambda \geq 0$. $\tilde{g}_a(\lambda)$ is an increasing function of λ for every $a \in B$, hence $\tilde{g}(\lambda)$ is an increasing function of λ . So \tilde{g} satisfies the conditions in lemma 1.1. Hence, there exist subordinate function $\epsilon(\lambda)$ and a constant $C' > 1$ such that $\tilde{g}(\lambda) \leq M(\epsilon(\lambda)) + \ln C'$. We get that

$$\ln_+ \left(\frac{|D_\xi^\alpha D_x^\beta D_y^\gamma a(x, y, \xi)| \langle (x, y, \xi) \rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}}{h^{|\alpha|+|\beta|+|\gamma|} \langle x-y \rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|} A_\alpha B_{\beta+\gamma}} \right) \leq \tilde{g}(|(x, y, \xi)|) \leq M(\epsilon(|(x, y, \xi)|)) + \ln C',$$

for all $(x, y, \xi) \in \mathbb{R}^{3d}$, $\alpha, \beta, \gamma \in \mathbb{N}^d$ and $a \in B$, i.e.

$$\frac{|D_\xi^\alpha D_x^\beta D_y^\gamma a(x, y, \xi)| \langle (x, y, \xi) \rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}}{h^{|\alpha|+|\beta|+|\gamma|} \langle x-y \rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|} A_\alpha B_{\beta+\gamma}} \leq C' e^{M(\epsilon(|(x, y, \xi)|))},$$

for all $(x, y, \xi) \in \mathbb{R}^{3d}$, $\alpha, \beta, \gamma \in \mathbb{N}^d$ and $a \in B$. By lemma 3.12 of [9], there exists another sequence \tilde{N}_p , which satisfies (M.1), such that $\tilde{N}(\lambda) \geq M(\epsilon(\lambda))$ and $k'_p = \tilde{n}_p/m_p \rightarrow \infty$ when $p \rightarrow \infty$. There exists $(k''_p) \in \mathfrak{R}$ such that $k''_p \leq k'_p$, for $p \in \mathbb{Z}_+$. Then

$$e^{N_{k''_p}(\lambda)} = \sup_p \frac{\lambda^p}{M_p \prod_{j=1}^p k''_j} \geq \sup_p \frac{\lambda^p}{M_p \prod_{j=1}^p k'_j} = \sup_p \frac{\lambda^p \tilde{N}_0}{\tilde{N}_p} = e^{\tilde{N}(\lambda)} \geq e^{M(\epsilon(\lambda))}.$$

From this, we obtain the estimate

$$\frac{|D_\xi^\alpha D_x^\beta D_y^\gamma a(x, y, \xi)| \langle (x, y, \xi) \rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}}{h^{|\alpha|+|\beta|+|\gamma|} \langle x-y \rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|} A_\alpha B_{\beta+\gamma}} \leq C e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)},$$

for all $(x, y, \xi) \in \mathbb{R}^{3d}$, $\alpha, \beta, \gamma \in \mathbb{N}^d$ and $a \in B$, where we choose $(k_p) \in \mathfrak{R}$ such that $e^{N_{k''_p}(|(x, y, \xi)|)} \leq c' e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)}$, for some constant $c' > 0$. \square

Lemma 2.2. *Let $a \in \Pi_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{3d})$. For $\delta > 0$ and $u, \chi \in \mathcal{S}^*(\mathbb{R}^d)$, such that $\chi(0) = 1$, define*

$$I_{\chi, \delta}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} a(x, y, \xi) \chi(\delta\xi) u(y) dy d\xi.$$

Then $I_{\chi, \delta}(x)$ has a limit when $\delta \rightarrow 0^+$ and the limit doesn't depend on χ . Moreover, the limit function is continuous and has ultrapolynomial growth of class $$.*

Proof. The (M_p) case. Let $a \in \Pi_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{3d}; m)$. For $l > 0$ consider $P_l(\xi)$. By proposition 1.1, $P_l(\xi)$ is never zero and we can choose $P_l(\xi)$ such that, $|P_l(\xi)| \geq c_1 e^{M(r|\xi|)}$, where $r > 0$ is such that $\int_{\mathbb{R}^d} e^{M(m|\xi|)} e^{-M(r|\xi|)} d\xi < \infty$. Also, we have the estimate $\left| D_\xi^\alpha \frac{1}{P_l(\xi)} \right| \leq c'_1 \frac{\alpha!}{d_1^{|\alpha|}}$, for some $c'_1 > 0$ and $d_1 > 0$. On the other hand if we represent $P_l(\xi) = \sum_\alpha c_\alpha \xi^\alpha$ then there exist $L_0 > 0$ and $C_0 > 0$ such that $|c_\alpha| \leq C_0 L_0^{|\alpha|} / M_\alpha$. Observe that

$$e^{i(x-y)\xi} = \frac{1}{P_l(y-x)} P_l(-D_\xi) \left(\frac{1}{P_l(\xi)} P_l(-D_y) e^{i(x-y)\xi} \right).$$

Then we have

$$I_{\chi, \delta}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} \frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) (a(x, y, \xi) \chi(\delta\xi) u(y)) \right) dy d\xi. \quad (5)$$

Because $u, \chi \in \mathcal{S}^{(M_p)}(\mathbb{R}^d)$, for every $s > 0$

$$\sup_{\alpha \in \mathbb{N}^d} \frac{s^{|\alpha|} \|e^{M(s|\xi|)} D^\alpha \chi(\xi)\|_{L^\infty}}{M_\alpha} < \infty, \quad \sup_{\alpha \in \mathbb{N}^d} \frac{s^{|\alpha|} \|e^{M(s|y|)} D^\alpha u(y)\|_{L^\infty}}{M_\alpha} < \infty.$$

Now, we estimate as follows

$$\begin{aligned} & \left| P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) (a(x, y, \xi) \chi(\delta\xi) u(y)) \right) \right| \\ & \leq \sum_{\alpha, \gamma} |c_\alpha| |c_\gamma| \sum_{\substack{\alpha' \leq \alpha \\ \gamma' \leq \gamma}} \sum_{\gamma'' \leq \gamma'} \binom{\alpha}{\alpha'} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} \\ & \quad \cdot \left| D_\xi^{\alpha'} D_y^{\gamma''} a(x, y, \xi) \right| \left| D_y^{\gamma' - \gamma''} \frac{1}{P_l(y-x)} \right| \delta^{|\alpha| - |\alpha'|} \left| D_\xi^{\alpha - \alpha'} \chi(\delta\xi) \right| \left| D_y^{\gamma - \gamma'} u(y) \right| \\ & \leq C' \sum_{\alpha, \gamma} |c_\alpha| |c_\gamma| \sum_{\substack{\alpha' \leq \alpha \\ \gamma' \leq \gamma}} \sum_{\gamma'' \leq \gamma'} \binom{\alpha}{\alpha'} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} \frac{(\gamma' - \gamma'')!}{d_1^{|\gamma' - \gamma''|}} \\ & \quad \cdot \frac{h^{|\alpha'| + |\gamma''|} \langle x - y \rangle^{\rho|\alpha'| + \rho|\gamma''|} A_{\alpha'} B_{\gamma''} e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{\langle (x, y, \xi) \rangle^{\rho|\alpha'| + \rho|\gamma''|}} \delta^{|\alpha| - |\alpha'|} \frac{M_{\alpha - \alpha'} M_{\gamma - \gamma'} e^{-M(s\delta|\xi|)}}{s^{|\alpha| - |\alpha'| + |\gamma| - |\gamma'|} e^{M(s|y|)}} \end{aligned}$$

$$\begin{aligned}
&\leq C''' \sum_{\alpha, \gamma} \frac{L_0^{|\alpha|+|\gamma|}}{M_\alpha M_\gamma} \sum_{\substack{\alpha' \leq \alpha \\ \gamma' \leq \gamma}} \sum_{\gamma'' \leq \gamma'} \binom{\alpha}{\alpha'} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} \frac{(\gamma' - \gamma'')! (4L_0)^{|\gamma'| - |\gamma''|}}{d_1^{|\gamma'| - |\gamma''|} (4L_0)^{|\gamma'| - |\gamma''|}} \\
&\quad \cdot \frac{(2Lh)^{|\alpha|+|\gamma'|} e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{(2Lh)^{|\gamma'| - |\gamma''|} M_{\gamma' - \gamma''}} \delta^{|\alpha| - |\alpha'|} \frac{M_\alpha M_\gamma}{s^{|\alpha| - |\alpha'| + |\gamma| - |\gamma'|} e^{M(s|y|)}} \\
&\leq C_1 \frac{e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{e^{M(s|y|)}} \\
&\quad \cdot \sum_{\alpha, \gamma} \left(\frac{\delta L_0}{s} \right)^{|\alpha|} \left(\frac{L_0}{s} \right)^{|\gamma|} \sum_{\substack{\alpha' \leq \alpha \\ \gamma' \leq \gamma}} \binom{\alpha}{\alpha'} \binom{\gamma}{\gamma'} \left(\frac{2sLh}{\delta} \right)^{|\alpha'|} (2sLh)^{|\gamma'|} \sum_{\gamma'' \leq \gamma'} \binom{\gamma'}{\gamma''} \frac{1}{(8L_0 Lh)^{|\gamma'| - |\gamma''|}} \\
&= C_1 \frac{e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{e^{M(s|y|)}} \sum_{\alpha, \gamma} \left(\frac{\delta L_0}{s} + 2LL_0h \right)^{|\alpha|} \left(\frac{L_0}{s} + 2L_0Lh + \frac{1}{4} \right)^{|\gamma|}.
\end{aligned}$$

Choose h such that $LL_0h < 1/8$ and then choose s such that the above sum converge for $\delta = 1$ and denote its value by C_2 (then, obviously, for $0 < \delta < 1$ the sum is not greater than C_2). Moreover, choose s large enough, such that $\int_{\mathbb{R}^d} e^{M(m|y|)} e^{-M(s|y|)} dy < \infty$. Hence

$$\begin{aligned}
|I_{\chi, \delta}(x)| &\leq \frac{C_1 C_2}{(2\pi)^d} \int_{\mathbb{R}^{2d}} \frac{1}{|P_l(\xi)|} \frac{e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{e^{M(s|y|)}} dy d\xi \\
&\leq C e^{M(m|x|)} \int_{\mathbb{R}^d} \frac{e^{M(m|\xi|)}}{e^{M(r|\xi|)}} d\xi \cdot \int_{\mathbb{R}^d} \frac{e^{M(m|y|)}}{e^{M(s|y|)}} dy,
\end{aligned}$$

which is finite for every x . Note that $a(x, y, \xi) \chi(\delta \xi) u(y) \rightarrow a(x, y, \xi) u(y)$ in $\mathcal{E}^{(M_p)}(\mathbb{R}_{y, \xi}^{2d})$ for each fixed x when $\delta \rightarrow 0^+$, so $\frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) (a(x, y, \xi) \chi(\delta \xi) u(y)) \right)$ tends to $\frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) a(x, y, \xi) u(y) \right)$ in $\mathcal{E}^{(M_p)}(\mathbb{R}_{y, \xi}^{2d})$ for each fixed x , when $\delta \rightarrow 0^+$. If we take the limit in (5) as $\delta \rightarrow 0^+$, from dominated convergence, it follows that

$$\lim_{\delta \rightarrow 0^+} I_{\chi, \delta}(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} \frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) (a(x, y, \xi) u(y)) \right) dy d\xi.$$

Moreover, by similar estimates as above, one proves that the function in the last integral can be dominated by $C e^{M(m|x|)} \frac{e^{M(m|\xi|)}}{e^{M(r|\xi|)}} \cdot \frac{e^{M(m|y|)}}{e^{M(s|y|)}}$. Thus, $\lim_{\delta \rightarrow 0^+} I_{\chi, \delta}(x)$ is a continuous function with (M_p) -ultrapolynomial growth. Note that the choice of P_l does not depend on χ and u , only on m such that $a \in \Pi_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{3d}; m)$. Hence, one can choose the same P_l for all $a \in \Pi_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{3d}; m)$. From this, the claim in the lemma follows.

The $\{M_p\}$ case. Let $a \in \Pi_{A_p, B_p}^{\{M_p\}, \infty}(\mathbb{R}^{3d}; h)$. By lemma 2.1 there exists $(k_p) \in \mathfrak{R}$ such

that

$$\left| D_\xi^\alpha D_x^\beta D_y^\gamma a(x, y, \xi) \right| \leq C_0 \frac{h^{|\alpha|+|\beta|+|\gamma|} \langle x-y \rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|} A_\alpha B_{\beta+\gamma} e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)}}{\langle (x, y, \xi) \rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}}, \quad (6)$$

for all $\alpha, \beta, \gamma \in \mathbb{N}^d$ and $(x, y, \xi) \in \mathbb{R}^{3d}$. For $(l_p) \in \mathfrak{R}$ consider $P_{l_p}(\xi)$. By proposition 1.1, we can choose $P_{l_p}(\xi)$ such that, $|P_{l_p}(\xi)| \geq c'' e^{N_{r_p}(|\xi|)}$, where $(r_p) \in \mathfrak{R}$ is such that $\int_{\mathbb{R}^d} e^{N_{k_p}(|\xi|)} e^{-N_{r_p}(|\xi|)} d\xi < \infty$. On the other hand, if we represent $P_{l_p}(\xi) = \sum_{\alpha} c_\alpha \xi^\alpha$, then for every $L' > 0$ there exists $C' > 0$ such that $|c_\alpha| \leq C' L'^{|\alpha|} / M_\alpha$. Also, we have the same estimates, as in the (M_p) case, for the derivatives of $1/P_{l_p}(\xi)$, i.e $\left| D_\xi^\alpha \frac{1}{P_{l_p}(\xi)} \right| \leq c'_1 \frac{\alpha!}{d_1^{|\alpha|}}$, for some $c'_1 > 0$ and $d_1 > 0$. Because $u, \chi \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$, there exists $s > 0$, such that

$$\sup_{\alpha \in \mathbb{N}^d} \frac{s^{|\alpha|} \|e^{M(s|\xi|)} D^\alpha \chi(\xi)\|_{L^\infty}}{M_\alpha} < \infty, \quad \sup_{\alpha \in \mathbb{N}^d} \frac{s^{|\alpha|} \|e^{M(s|y|)} D^\alpha u(y)\|_{L^\infty}}{M_\alpha} < \infty.$$

(We can choose s to be the same for u and χ). Similarly as for the (M_p) case, one obtains (5), but with P_{l_p} in place of P_l and obtains the estimate

$$\begin{aligned} & \left| P_{l_p}(D_y) \left(\frac{1}{P_{l_p}(y-x)} P_{l_p}(D_\xi) (a(x, y, \xi) \chi(\delta \xi) u(y)) \right) \right| \\ & \leq C_1 \frac{e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)}}{e^{M(s|y|)}} \sum_{\alpha, \gamma} \left(\frac{\delta L'}{s} + 2LL'h \right)^{|\alpha|} \left(\frac{L'}{s} + 2L'Lh + \frac{1}{4} \right)^{|\gamma|}. \end{aligned}$$

Choose L' , small enough, such that the above sum converges for $\delta = 1$ and denote its value by C_2 . Similarly as above, we obtain the estimate

$$|I_{\chi, \delta}(x)| \leq C e^{N_{k_p}(|x|)} \int_{\mathbb{R}^d} e^{-N_{r_p}(|\xi|)} e^{N_{k_p}(|\xi|)} d\xi \cdot \int_{\mathbb{R}^d} e^{N_{k_p}(|y|)} e^{-M(s|y|)} dy.$$

The first integral converges by the choice of (r_p) and the convergence of the second can be easily proven. By similar arguments as in the (M_p) case and dominated convergence, the claim of the lemma follows. Note that the choice of P_{l_p} does not depend on u and χ , only on a . \square

By the lemma, $\lim_{\delta \rightarrow 0^+} I_{\chi, \delta}(x)$ is an element of $\mathcal{S}'^*(\mathbb{R}^d)$. For every $a \in \Pi_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{3d})$ define the operator $A : \mathcal{S}'^*(\mathbb{R}^d) \rightarrow \mathcal{S}'^*(\mathbb{R}^d)$, $Au(x) = \lim_{\delta \rightarrow 0^+} I_{\chi, \delta}(x)$. By the proof of the above lemma we obtain that

$$Au(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} \frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) a(x, y, \xi) u(y) \right) dy d\xi,$$

for the (M_p) case, respectively

$$Au(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} \frac{1}{P_{l_p}(\xi)} P_{l_p}(D_y) \left(\frac{1}{P_{l_p}(y-x)} P_{l_p}(D_\xi) a(x, y, \xi) u(y) \right) dy d\xi,$$

for the $\{M_p\}$ case and moreover, the choice of P_l in the (M_p) case, respectively P_{l_p} in the $\{M_p\}$ case does not depend on $u \in \mathcal{S}^*$. If $P_{l'}$, resp. $P_{l'_p}$, is another operator such that $|P_{l'}(\xi)| \geq c_1 e^{M(r|\xi|)}$, resp. $|P_{l'_p}(\xi)| \geq c'' e^{N_{r_p}(|\xi|)}$, where $\int_{\mathbb{R}^d} e^{M(m|\xi|)} e^{-M(r|\xi|)} d\xi < \infty$, resp. $\int_{\mathbb{R}^d} e^{N_{k_p}(|\xi|)} e^{-N_{r_p}(|\xi|)} d\xi < \infty$, then $Au(x)$ can be given in the above form with $P_{l'}$ in place of P_l , resp. $P_{l'_p}$ in place of P_{l_p} . To prove the continuity of A , put

$$K(x, y, \xi) = e^{i(x-y)\xi} \frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) a(x, y, \xi) u(y) \right)$$

in the (M_p) case, resp., the same but with P_{l_p} in place of P_l , in the $\{M_p\}$ case. For $v \in \mathcal{S}^*$,

$$\langle Au(x), v(x) \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{3d}} K(x, y, \xi) v(x) dy d\xi dx.$$

Let $v \in \mathcal{S}^*$ be fixed. If $u \in B$, where B is a bounded set in \mathcal{S}^* , similarly as in the proof of the above lemma, one can prove that $\langle Au(x), v(x) \rangle$ is bounded when $u \in B$. Hence the set $A(B)$ is simply bounded in \mathcal{S}'^* , consequently it is strongly bounded. Because \mathcal{S}^* is bornological and A maps bounded sets into bounded sets it must be continuous.

Theorem 2.2. *The mapping $(a, u) \mapsto Au$, $\Pi_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{3d}) \times \mathcal{S}^*(\mathbb{R}^d) \longrightarrow \mathcal{S}^*(\mathbb{R}^d)$, is hypocontinuous.*

Proof. Because $\Pi_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{3d})$ and $\mathcal{S}^*(\mathbb{R}^d)$ are barreled it is enough to prove that the mapping is separately continuous. We will consider first the (M_p) case. It is enough to prove that, for every $m > 0$, the mapping $\Pi_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{3d}; m) \times \mathcal{S}^{(M_p)}(\mathbb{R}^d) \longrightarrow \mathcal{S}^{(M_p)}(\mathbb{R}^d)$ is separately continuous. We will prove that it is continuous i.e. that for every $s > 0$, there exists a constant $C > 0$ and $h > 0$, $t > 0$ such that $\|Au\|_s \leq C \|a\|_{h, m, \Pi} \|u\|_t$, where $\|\phi\|_s = \sup_{\alpha} \frac{s^{|\alpha|} \|D^\alpha \phi(\cdot) e^{M(s|\cdot|)}\|_{L^\infty}}{M_\alpha}$ are the seminorms in $\mathcal{S}^{(M_p)}(\mathbb{R}^d)$. Let $s > 0$. Obviously, without losing generality, we can assume that $s \geq 1$. Choose $P_l(\xi)$ as in the proof of the above lemma and represent Au in the form

$$Au(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} \frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) a(x, y, \xi) u(y) \right) dy d\xi.$$

In the proof of the above lemma we proved that P_l can be chosen the same for all $a \in \Pi_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{3d}; m)$ (it depends only on m). By proposition 1.1 $P_l(\xi)$ is never zero and we can

choose l small enough such that $\left| D_\xi^\alpha \frac{1}{P_l(\xi)} \right| \leq c'_1 \frac{\alpha!}{d_1^{|\alpha|}} e^{-M(r|\xi|)}$, for some $c'_1 > 0$ and $d_1 > 0$, where $r > 0$ is such that $\int_{\mathbb{R}^d} \frac{e^{M(m|\xi|)} e^{M(2s|\xi|)}}{e^{M(r|\xi|)}} d\xi$ converges and $e^{M(\frac{r}{2}|x|)} \geq \tilde{C} e^{M(s|x|)} e^{M(m|x|)}$. On the other hand, if we represent $P_l(\xi) = \sum_{\alpha} c_\alpha \xi^\alpha$, there exist $L_0 \geq 1$ and $C_0 > 0$ such that $|c_\alpha| \leq C_0 L_0^{|\alpha|} / M_\alpha$. Then, for $a \in \Pi_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{3d}, m)$ and $u \in \mathcal{S}^{(M_p)}$, we have

$$\left| D_x^\beta \left(e^{i(x-y)\xi} \frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) a(x, y, \xi) u(y) \right) \right) \right|$$

$$\begin{aligned} &\leq \sum_{\beta' \leq \beta} \sum_{\alpha, \gamma} \sum_{\gamma' \leq \gamma} \sum_{\gamma'' \leq \gamma'} \sum_{\beta'' \leq \beta'} \binom{\beta}{\beta'} \binom{\beta'}{\beta''} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} |c_\alpha| |c_\gamma| \frac{|\xi|^{|\beta| - |\beta'|}}{|P_l(\xi)|} \\ &\quad \cdot \left| D_y^{\gamma' - \gamma''} D_x^{\beta' - \beta''} \left(\frac{1}{P_l(y-x)} \right) \right| \left| D_\xi^\alpha D_x^{\beta''} D_y^{\gamma''} a(x, y, \xi) \right| \left| D_y^{\gamma - \gamma'} u(y) \right| \\ &\leq C_1 \|a\|_{h, m, \Pi} \|u\|_t \sum_{\beta' \leq \beta} \sum_{\alpha, \gamma} \sum_{\gamma' \leq \gamma} \sum_{\gamma'' \leq \gamma'} \sum_{\beta'' \leq \beta'} \binom{\beta}{\beta'} \binom{\beta'}{\beta''} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} \frac{L_0^{|\alpha| + |\gamma|}}{M_\alpha M_\gamma} \frac{|\xi|^{|\beta| - |\beta'|}}{e^{M(r|\xi|)}} \\ &\quad \cdot \frac{(\beta' - \beta'' + \gamma' - \gamma'')! e^{-M(r|x-y|)}}{d_1^{|\beta'| - |\beta''| + |\gamma'| - |\gamma''|}} \cdot \frac{M_{\gamma - \gamma'} e^{-M(t|y|)}}{t^{|\gamma| - |\gamma'|}} \\ &\quad \cdot \frac{h^{|\alpha| + |\beta''| + |\gamma''|} \langle x - y \rangle^{\rho|\alpha| + \rho|\beta''| + \rho|\gamma''|} A_\alpha B_{\beta'' + \gamma''} e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{\langle (x, y, \xi) \rangle^{\rho|\alpha| + \rho|\beta''| + \rho|\gamma''|}} \\ &\leq C_2 \|a\|_{h, m, \Pi} \|u\|_t \sum_{\beta' \leq \beta} \sum_{\alpha, \gamma} \sum_{\gamma' \leq \gamma} \sum_{\gamma'' \leq \gamma'} \sum_{\beta'' \leq \beta'} \binom{\beta}{\beta'} \binom{\beta'}{\beta''} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} L_0^{|\alpha| + |\gamma|} \frac{|\xi|^{|\beta| - |\beta'|}}{e^{M(r|\xi|)}} \\ &\quad \cdot \frac{(\beta' - \beta'' + \gamma' - \gamma'')! e^{-M(r|x-y|)}}{d_1^{|\beta'| - |\beta''| + |\gamma'| - |\gamma''|}} \cdot \frac{e^{-M(t|y|)}}{t^{|\gamma| - |\gamma'|} M_{\gamma' - \gamma''}} \\ &\quad \cdot (2hL)^{|\alpha| + |\beta''| + |\gamma''|} H^{|\beta''| + |\gamma''|} M_{\beta''} e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m|y|)} \\ &\leq C_2 \|a\|_{h, m, \Pi} \|u\|_t \frac{M_\beta e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{e^{M(r|\xi|)} e^{M(t|y|)} e^{M(r|x-y|)}} \\ &\quad \cdot \sum_{\beta' \leq \beta} \sum_{\alpha, \gamma} \sum_{\gamma' \leq \gamma} \sum_{\gamma'' \leq \gamma'} \sum_{\beta'' \leq \beta'} \binom{\beta}{\beta'} \binom{\beta'}{\beta''} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} L_0^{|\alpha| + |\gamma|} \frac{|\xi|^{|\beta| - |\beta'|}}{M_{\beta - \beta'}} \cdot \frac{(2s)^{|\beta| - |\beta'|}}{(2s)^{|\beta| - |\beta'|}} \\ &\quad \cdot \frac{(\beta' - \beta'' + \gamma' - \gamma'')!}{d_1^{|\beta'| - |\beta''| + |\gamma'| - |\gamma''|}} \cdot \frac{(2hL)^{|\alpha| + |\beta''| + |\gamma''|} H^{|\beta''| + |\gamma''|}}{t^{|\gamma| - |\gamma'|} M_{\beta' - \beta''} M_{\gamma' - \gamma''}} \\ &\leq C_3 \|a\|_{h, m, \Pi} \|u\|_t \frac{M_\beta e^{M(m|\xi|)} e^{M(2s|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{e^{M(r|\xi|)} e^{M(t|y|)} e^{M(r|x-y|)}} \\ &\quad \cdot \sum_{\beta' \leq \beta} \sum_{\alpha, \gamma} \sum_{\gamma' \leq \gamma} \sum_{\gamma'' \leq \gamma'} \sum_{\beta'' \leq \beta'} \binom{\beta}{\beta'} \binom{\beta'}{\beta''} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} (2hLL_0)^{|\alpha|} L_0^{|\gamma|} \end{aligned}$$

$$\begin{aligned}
& \cdot \frac{(\beta' - \beta'' + \gamma' - \gamma'')!(4sHL_0)^{|\beta'| - |\beta''| + |\gamma'| - |\gamma''|}}{d_1^{|\beta'| - |\beta''| + |\gamma'| - |\gamma''|} M_{\beta' - \beta'' + \gamma' - \gamma''}(4sHL_0)^{|\beta'| - |\beta''| + |\gamma'| - |\gamma''|}} \cdot \frac{(2hL)^{|\beta''| + |\gamma''|} H^{|\beta'| + |\gamma'|}}{t^{|\gamma| - |\gamma'|} (2s)^{|\beta| - |\beta'|}} \\
& \leq C_4 \|a\|_{h,m,\Pi} \|u\|_t \frac{M_\beta e^{M(m|\xi|)} e^{M(2s|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{e^{M(r|\xi|)} e^{M(t|y|)} e^{M(r|x-y|)}} \\
& \quad \cdot \sum_{\beta' \leq \beta} \sum_{\alpha, \gamma} \sum_{\gamma' \leq \gamma} \sum_{\gamma'' \leq \gamma'} \sum_{\beta'' \leq \beta'} \binom{\beta}{\beta'} \binom{\beta'}{\beta''} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} (2hLL_0)^{|\alpha|} L_0^{|\gamma|} \\
& \quad \cdot \frac{(2hL)^{|\beta''| + |\gamma''|} H^{|\beta'| + |\gamma'|}}{(4sHL_0)^{|\beta'| - |\beta''| + |\gamma'| - |\gamma''|} t^{|\gamma| - |\gamma'|} (2s)^{|\beta| - |\beta'|}} \\
& = C_4 \|a\|_{h,m,\Pi} \|u\|_t \frac{M_\beta e^{M(m|\xi|)} e^{M(2s|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{e^{M(r|\xi|)} e^{M(t|y|)} e^{M(r|x-y|)}} \\
& \quad \cdot \left(\frac{1}{2s} + 2hLH + \frac{1}{4sL_0} \right)^{|\beta|} \sum_{\alpha, \gamma} (2hLL_0)^{|\alpha|} \left(\frac{L_0}{t} + 2hLL_0H + \frac{1}{4s} \right)^{|\gamma|}.
\end{aligned}$$

Note that $e^{M(\frac{r}{2}|x|)} \leq C_6 e^{M(r|x-y|)} e^{M(r|y|)}$. For the chosen r we choose t such that the integral $\int_{\mathbb{R}^d} \frac{e^{M(m|y|)} e^{M(r|y|)}}{e^{M(t|y|)}} dy$ converges and moreover, we take h small enough and t large enough such that the above sum converges. Moreover, choose h small enough such that $\frac{1}{2s} + 2hLH + \frac{1}{4sL_0} \leq \frac{1}{s}$. Then for the derivatives of Au we obtain

$$|D_x^\beta Au(x)| \leq C \|a\|_{h,m,\Pi} \|u\|_t e^{-M(s|x|)} \frac{M_\beta}{s^{|\beta|}},$$

which is the desired estimate.

Now we will consider the $\{M_p\}$ case. Note that it is enough to prove that, for every $h > 0$, $\Pi_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{3d}; h) \times \mathcal{S}^{\{M_p\}}(\mathbb{R}^d) \longrightarrow \mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ is separately continuous. Because $\Pi_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{3d}; h)$ and $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ are bornological it is enough to prove that this mapping maps products of bounded sets into bounded sets in $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$. Let B_1 and B_2 be bounded sets in $\Pi_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{3d}; h)$, respectively in $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$. Then, by lemma 2.1, there exist $\tilde{C}_1 > 0$ and $(k_p) \in \mathfrak{R}$ such that

$$\frac{|D_\xi^\alpha D_x^\beta D_y^\gamma a(x, y, \xi)| \langle (x, y, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + \rho|\gamma|} e^{-N_{k_p}(|\xi|)} e^{-N_{k_p}(|x|)} e^{-N_{k_p}(|y|)}}{h^{|\alpha| + |\beta| + |\gamma|} \langle x - y \rangle^{\rho|\alpha| + \rho|\beta| + \rho|\gamma|} A_\alpha B_{\beta + \gamma}} \leq \tilde{C}_1, \quad (7)$$

for all $a \in B_1$, $(x, y, \xi) \in \mathbb{R}^{3d}$ and $\alpha, \beta, \gamma \in \mathbb{N}^d$. We know that $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d) = \varinjlim_{s \rightarrow 0} \mathcal{S}_\infty^{M_p, s}(\mathbb{R}^d)$,

where $\mathcal{S}_\infty^{M_p, s}(\mathbb{R}^d)$ is the (B) - space with the norm $\|\phi\|_s = \sup_\alpha \frac{s^{|\alpha|} \|D^\alpha \phi(\cdot) e^{M(s|\cdot|)}\|_{L^\infty}}{M_\alpha}$ and $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d)$ is a (DFS) - space generated by this inductive limit (the linking mappings are compact inclusions). So, there exists $t > 0$ such that $B_2 \subseteq \mathcal{S}_\infty^{M_p, t}(\mathbb{R}^d)$ and it is bounded there. Hence, there exists $\tilde{C}_2 > 0$ such that $\|u\|_t \leq \tilde{C}_2$, for all $u \in B_2$. On the other

hand, we know that $\mathcal{S}^{\{M_p\}}(\mathbb{R}^d) = \lim_{(s_p), (s'_p) \in \mathfrak{R}} \mathcal{S}_{(s_p), (s'_p)}^{M_p}(\mathbb{R}^d)$, where $\mathcal{S}_{(s_p), (s'_p)}^{M_p}(\mathbb{R}^d)$ is the (B) - space with the norm $\|\phi\|_{(s_p), (s'_p)} = \sup_{\alpha} \frac{\|D^\alpha \phi(\cdot) e^{N_{s'_p}(|\cdot|)}\|_{L^\infty}}{M_\alpha \prod_{j=1}^{|\alpha|} s_j}$. So, it is enough to prove that, for arbitrary $(s_p), (s'_p) \in \mathfrak{R}$, $\|Au\|_{(s_p), (s'_p)}$ is bounded for all $a \in B_1$ and $u \in B_2$. So, let $(s_p), (s'_p) \in \mathfrak{R}$ be fixed. Represent Au in the form

$$Au(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} \frac{1}{P_{l_p}(\xi)} P_{l_p}(D_y) \left(\frac{1}{P_{l_p}(y-x)} P_{l_p}(D_\xi) a(x, y, \xi) u(y) \right) dy d\xi.$$

In the proof of lemma 2.2 we proved that the choice of P_{l_p} depends only on (k_p) such that (7) holds. But (k_p) is the same for all $a \in B_1$ hence we can choose P_{l_p} the same for all $a \in B_1$. By proposition 1.1, $P_{l_p}(\xi)$ is never zero and we can choose $(l_p) \in \mathfrak{R}$ such that, $\left| D_\xi^\alpha \frac{1}{P_{l_p}(\xi)} \right| \leq c'_1 \frac{\alpha!}{d_1^{|\alpha|}} e^{-N_{r_p}(|\xi|)}$, for some $c'_1 > 0$ and $d_1 > 0$, where $(r_p) \in \mathfrak{R}$ is chosen such that $\int_{\mathbb{R}^d} \frac{e^{N_{k_p}(|\xi|)} e^{N_{s_p}(|\xi|)}}{e^{N_{r_p}(|\xi|)}} d\xi$ converges and $e^{N_{2r_p}(|x|)} \geq \tilde{C} e^{N_{s'_p}(|x|)} e^{N_{k_p}(|x|)}$ (see also the remarks after the proof of lemma 2.2). On the other hand, if we represent $P_{l_p}(\xi) = \sum_{\alpha} c_\alpha \xi^\alpha$, then

for every $L' > 0$ there exists $C' > 0$ such that $|c_\alpha| \leq C' L'^{|\alpha|} / M_\alpha$. For $a \in B_1$ and $u \in B_2$, similarly as in the (M_p) case, one obtains the estimate

$$\begin{aligned} & \left| D_x^\beta \left(e^{i(x-y)\xi} \frac{1}{P_{l_p}(\xi)} P_{l_p}(D_y) \left(\frac{1}{P_{l_p}(y-x)} P_{l_p}(D_\xi) a(x, y, \xi) u(y) \right) \right) \right| \\ & \leq C_2 \tilde{C}_1 \tilde{C}_2 \frac{M_\beta e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)}}{e^{N_{r_p}(|\xi|)} e^{M(t|y|)} e^{N_{r_p}(|x-y|)}} \\ & \quad \cdot \sum_{\beta' \leq \beta} \sum_{\alpha, \gamma} \sum_{\gamma' \leq \gamma} \sum_{\gamma'' \leq \gamma'} \sum_{\beta'' \leq \beta'} \binom{\beta}{\beta'} \binom{\beta'}{\beta''} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} L'^{|\alpha|+|\gamma|} \frac{|\xi|^{|\beta|-|\beta'|}}{M_{\beta-\beta'}} \cdot \prod_{j=1}^{|\beta|-|\beta'|} \frac{1}{s_j} \cdot \prod_{j=1}^{|\beta|-|\beta'|} s_j \\ & \quad \cdot \frac{(\beta' - \beta'' + \gamma' - \gamma'')!}{d_1^{|\beta'-\beta''|+|\gamma'-\gamma''|}} \cdot \frac{(2hL)^{|\alpha|+|\beta''|+|\gamma''|} H^{|\beta''|+|\gamma''|}}{t^{|\gamma|-|\gamma'|} M_{\beta'-\beta''} M_{\gamma'-\gamma''}} \\ & \leq C_3 \tilde{C}_1 \tilde{C}_2 \frac{M_\beta e^{N_{k_p}(|\xi|)} e^{N_{s_p}(|\xi|)} e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)}}{e^{N_{r_p}(|\xi|)} e^{M(t|y|)} e^{N_{r_p}(|x-y|)}} \\ & \quad \cdot \sum_{\beta' \leq \beta} \sum_{\alpha, \gamma} \sum_{\gamma' \leq \gamma} \sum_{\gamma'' \leq \gamma'} \sum_{\beta'' \leq \beta'} \binom{\beta}{\beta'} \binom{\beta'}{\beta''} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} (2hLL')^{|\alpha|} L'^{|\gamma|} \\ & \quad \cdot \frac{(\beta' - \beta'' + \gamma' - \gamma'')!}{d_1^{|\beta'-\beta''|+|\gamma'-\gamma''|} M_{\beta'-\beta''+\gamma'-\gamma''}} \cdot \frac{(2hL)^{|\beta''|+|\gamma''|} H^{|\beta''|+|\gamma''|}}{t^{|\gamma|-|\gamma'|}} \cdot \prod_{j=1}^{|\beta|-|\beta'|} s_j \\ & \leq C_4 \tilde{C}_1 \tilde{C}_2 \frac{M_\beta e^{N_{k_p}(|\xi|)} e^{N_{s_p}(|\xi|)} e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)}}{e^{N_{r_p}(|\xi|)} e^{M(t|y|)} e^{N_{r_p}(|x-y|)}} \end{aligned}$$

$$\begin{aligned}
& \cdot \sum_{\beta' \leq \beta} \sum_{\alpha, \gamma} \sum_{\gamma' \leq \gamma} \sum_{\gamma'' \leq \gamma'} \sum_{\beta'' \leq \beta'} \binom{\beta}{\beta'} \binom{\beta'}{\beta''} \binom{\gamma}{\gamma'} \binom{\gamma'}{\gamma''} (2hLL')^{|\alpha|} L'^{|\gamma|} \\
& \cdot \frac{(2hL)^{|\beta''|+|\gamma''|} H^{|\beta'|+|\gamma'|}}{t^{|\gamma|-|\gamma'|}} \cdot \prod_{j=1}^{|\beta|-|\beta'|} s_j \\
& \leq C_5 \tilde{C}_1 \tilde{C}_2 \frac{M_\beta e^{N_{k_p}(|\xi|)} e^{N_{s_p}(|\xi|)} e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)}}{e^{N_{r_p}(|\xi|)} e^{M(t|y|)} e^{N_{r_p}(|x-y|)}} \\
& \cdot 2^{|\beta|} \prod_{j=1}^{|\beta|} s_j \sum_{\alpha, \gamma} (2hLL')^{|\alpha|} \left(\frac{L'}{t} + 2hLL'H + L'H \right)^{|\gamma|},
\end{aligned}$$

where, in the last inequality, we used that $\frac{\lambda^p}{\prod_{j=1}^p s_j} \rightarrow 0$, when $p \rightarrow \infty$, for any fixed $\lambda > 0$ (i.e. it is bounded for all $p \in \mathbb{Z}_+$). (This follows from the fact that $(s_p) \in \mathfrak{R}$.) Note that $e^{N_{2r_p}(|x|)} \leq C_6 e^{N_{r_p}(|x-y|)} e^{N_{r_p}(|y|)}$. Also, the integral $\int_{\mathbb{R}^d} \frac{e^{N_{k_p}(|y|)} e^{N_{r_p}(|y|)}}{e^{M(t|y|)}} dy$ converges (this easily follows from the fact that $e^{N_{k_p}(|y|)} \leq c'' e^{M(t'|y|)}$ for every $t' > 0$, where the constant c'' depends on t' ; similarly for $e^{N_{r_p}(|y|)}$). Take L' such that the sum converges. Then, for the derivatives of Au , we obtain $|D_x^\beta Au(x)| \leq C \tilde{C}_1 \tilde{C}_2 e^{-N_{s_p}(|x|)} M_\beta \prod_{j=1}^{|\beta|} (2s_j)$, i.e.

$$\|Au\|_{(2s_p), (s'_p)} \leq C \tilde{C}_1 \tilde{C}_2, \text{ for all } a \in B_1 \text{ and } u \in B_2. \quad \square$$

Let $\tau \in \mathbb{R}$ be fixed. The inclusion $\Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d}) \rightarrow \Pi_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{3d})$, $b \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$, $b \mapsto a$, where $a(x, y, \xi) = b((1-\tau)x + \tau y, \xi)$, is continuous. Moreover, if $u, \phi \in \mathcal{S}^*(\mathbb{R}^d)$ such that $\phi(0) = 1$, by theorem 2.1, we have

$$\begin{aligned}
\text{Op}_\tau(b)u(x) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} b((1-\tau)x + \tau y, \xi) u(y) dy d\xi \\
&= \lim_{\delta \rightarrow 0^+} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} b((1-\tau)x + \tau y, \xi) \phi(\delta \xi) u(y) dy d\xi.
\end{aligned}$$

Hence, the operator $\text{Op}_\tau(b)$ coincides with the operator B corresponding to b when we observe $b((1-\tau)x + \tau y, \xi)$ as an element of $\Pi_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{3d})$. We get that the mapping $(b, u) \mapsto \text{Op}_\tau(b)u$, $\Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d}) \times \mathcal{S}^*(\mathbb{R}^d) \rightarrow \mathcal{S}^*(\mathbb{R}^d)$, is hypocontinuous. For $b \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$, denote its kernel by $K(x, y)$. If we consider the transposed of the operator $\text{Op}_\tau(b)$ then its kernel is $K(y, x)$. On the other hand, by (1),

$$K(y, x) = \mathcal{F}_{\xi \rightarrow x-y}^{-1}(b)(\tau x + (1-\tau)y, -\xi).$$

Hence ${}^t\text{Op}_\tau(b(x, \xi)) = \text{Op}_{1-\tau}(b(x, -\xi))$ i.e. ${}^t\text{Op}_\tau(b)$ is pseudo-differential operator and by the above it is a continuous mapping from $\mathcal{S}^*(\mathbb{R}^d)$ to $\mathcal{S}^*(\mathbb{R}^d)$. Using this we can extend $\text{Op}_\tau(b)$ to a continuous operator from $\mathcal{S}'^*(\mathbb{R}^d)$ to $\mathcal{S}'^*(\mathbb{R}^d)$ in the following way

$$\langle \text{Op}_\tau(b)u, v \rangle = \langle u, {}^t\text{Op}_\tau(b)v \rangle, \quad u \in \mathcal{S}'^*(\mathbb{R}^d), v \in \mathcal{S}^*(\mathbb{R}^d).$$

We need the following technical lemmas.

Lemma 2.3. *Let M_p be a sequence which satisfies (M.1), (M.2) and (M.3) and m a positive real. Then, for all $n \in \mathbb{Z}_+$, $M(mm_n) \leq 2(c_0m + 2)n \ln H + \ln c_0$, where c_0 is the constant from the conditions (M.2) and (M.3). If $(t_p) \in \mathfrak{R}$ then, $N_{t_p}(mm_n) \leq n \ln H + \ln c$ for all $n \in \mathbb{Z}_+$, where the constant c depends only on M_p , (t_p) and m , but not on n .*

Proof. By (M.3), for all $p \geq n + 1$, $p \in \mathbb{N}$, we have

$$\frac{1}{m_{n+1}} + \frac{1}{m_{n+2}} + \dots + \frac{1}{m_p} \leq c_0 \frac{n}{m_{n+1}} \leq c_0 \frac{n}{m_n}.$$

If we multiply the above inequality with m_p and use the fact that the sequence m_n is monotonically increasing, we obtain $p - n \leq c_0 \frac{nm_p}{m_n}$, i.e. $\frac{mm_n}{m_p} \leq c_0 \frac{mn}{p - n}$. Hence, for $p \geq [c_0m]n + 2n \geq n + 1$, we obtain that $mm_n \leq m_p$. Denote by k the term $[c_0m] + 2$. $M(\rho)$ is monotonically increasing, so $M(mm_n) \leq M(m_{kn})$. For $p \geq kn$, we have

$$\frac{m_{kn}^{p+1}}{M_{p+1}} = \frac{m_{kn}^p}{M_p} \cdot \frac{m_{kn}}{m_{p+1}} \leq \frac{m_{kn}^p}{M_p}.$$

Hence $M(m_{kn}) = \sup_p \ln_+ \frac{m_{kn}^p}{M_p} = \sup_{p \leq kn} \ln_+ \frac{m_{kn}^p}{M_p}$. But, for $p \leq kn$, $p \in \mathbb{N}$, we have

$$\frac{m_{kn}^p}{M_p} \leq \frac{m_{kn+1} \cdot m_{kn+2} \cdot \dots \cdot m_{kn+p}}{M_p} = \frac{M_{kn+p}}{M_p M_{kn}} \leq c_0 H^{kn+p} \leq c_0 H^{2kn},$$

where, in the second inequality, we used (M.2). We obtained

$$M(mm_n) \leq M(m_{kn}) = \sup_{p \leq kn} \ln_+ \frac{m_{kn}^p}{M_p} \leq 2kn \ln H + \ln c_0 \leq 2(c_0m + 2)n \ln H + \ln c_0,$$

which completes the proof for the first part. For the second part, denote by T_p the product $\prod_{j=1}^p t_j$. Observe that, for $p \in \mathbb{Z}_+$, we have

$$\frac{m^p m_n^p}{T_p M_p} \leq \frac{m^p m_{n+1} \cdot m_{n+2} \cdot \dots \cdot m_{n+p}}{T_p M_p} = \frac{m^p M_{n+p}}{T_p M_p M_n} \leq c_0 H^n \frac{(mH)^p}{T_p} \leq c H^n,$$

where, in the last inequality, we used the fact that (t_p) monotonically increases to infinity. Obviously c does not depend on p or n , only on m , (t_p) and M_p . From this we obtain $N_{t_p}(mm_n) \leq n \ln H + \ln c$, which completes the second part of the lemma. \square

Lemma 2.4. *Let M_p be a sequence which satisfies (M.1) and (M.3)' and $R > 1 + \frac{1}{M_1}$ be arbitrary. There exist a sequence $\psi_n(\xi) \in \mathcal{D}^*(\mathbb{R}^d)$, $n \in \mathbb{N}$, such that $\sum_{n=0}^{\infty} \psi_n = 1$, $\text{supp } \psi_0 \subseteq$*

$\{\xi \in \mathbb{R}^d | \langle \xi \rangle < 3RM_1\}$, $\text{supp } \psi_n \subseteq \{\xi \in \mathbb{R}^d | 2Rm_n < \langle \xi \rangle < 3Rm_{n+1}\}$, for $n \in \mathbb{Z}_+$ and for every $h > 0$ there exists $C > 0$, resp. there exist $h > 0$ and $C > 0$ such that

$$|D^\alpha \psi_0(\xi)| \leq C \left(\frac{h}{RM_1} \right)^{|\alpha|} M_\alpha, \text{ and } |D^\alpha \psi_n(\xi)| \leq C \left(\frac{h}{Rm_n} \right)^{|\alpha|} M_\alpha, \forall n \in \mathbb{Z}_+,$$

for all $\xi \in \mathbb{R}^d$ and $\alpha \in \mathbb{N}^d$.

Proof. Let $\phi \in \mathcal{D}^*$ such that $0 \leq \phi \leq 1$, $\phi(\xi) = 1$, for $\langle \xi \rangle < \sqrt{6}$, $\phi(\xi) = 0$, for $\langle \xi \rangle > 3$. Put

$$\psi_0(\xi) = \phi \left(\frac{\xi}{RM_1} \right), \psi_n(\xi) = \phi \left(\frac{\xi}{Rm_{n+1}} \right) - \phi \left(\frac{\xi}{Rm_n} \right).$$

It is easy to check that ψ_n , $n \in \mathbb{N}$, satisfy the claim in the lemma. \square

Let $\rho_0 = \inf\{\rho \in \mathbb{R}_+ | A_p \subset M_p^\rho\}$. Obviously $0 < \rho_0 \leq 1$. In general, the infimum can not be reached.

Counterexample. Let $r_1 = 1$ and $r_p = p^{1-1/(2\sqrt{\ln p})}$ for $p \in \mathbb{N}$, $p \geq 2$. The sequences r_p and $p^{1/(2\sqrt{\ln p})}$ are monotonically increasing. Put $R_p = \prod_{j=1}^p r_j$. Take $M_0 = 1$, $M_p = p!^2 R_p$ and $A_p = p!^2$. Then, obviously, A_p satisfies (M.1), (M.2) and (M.3). One easily checks that M_p satisfies (M.1), (M.2) and (M.3). It is clear that $A_p \subset M_p$. Note that $A_p \not\subset M_p^{2/3}$. In the contrary, there will exist $C > 0$ and $L > 0$ such that $p!^2 \leq CL^p p!^{4/3} R_p^{2/3}$, i.e. $\frac{p!}{L^{3p/2} R_p} \leq C_1$, for all $p \in \mathbb{Z}_+$, where we put $C_1 = C^{3/2}$. This is impossible, because this

means that $\sum_{j=1}^p \ln \frac{j}{L^{3/2} r_j}$ is bounded from above for all $p \in \mathbb{Z}_+$, but

$$\lim_{j \rightarrow \infty} \ln \frac{j}{L^{3/2} r_j} = \lim_{j \rightarrow \infty} \ln \frac{j^{1/(2\sqrt{\ln j})}}{L^{3/2}} = \lim_{j \rightarrow \infty} \left(\frac{\sqrt{\ln j}}{2} - \frac{3}{2} \ln L \right) = \infty.$$

On the other hand, note that for $\lambda > 2/3$, $A_p \subset M_p^\lambda$. This is true because

$$\frac{p!^2}{p!^{2\lambda} R_p^\lambda} = \frac{p!^{2(1-\lambda)}}{R_p^\lambda} = \prod_{j=2}^p \frac{j^{2(1-\lambda)}}{j^{\lambda - \lambda/(2\sqrt{\ln j})}} = \prod_{j=2}^p \frac{j^{\lambda/(2\sqrt{\ln j})}}{j^{3\lambda-2}}$$

and the last term converges to zero when $p \rightarrow \infty$ (note that $3\lambda - 2 > 0$ when $\lambda > 2/3$). From now on we will assume that ρ is such that $\rho_0 \leq \rho \leq 1$ if the infimum can be reached, otherwise $\rho_0 < \rho \leq 1$.

For $0 < r < 1$, define the set $\Omega_r = \{(x, y) \in \mathbb{R}^{2d} | |x - y| > r\langle x \rangle\}$.

Lemma 2.5. *Let $0 < r < 1$. There exists $\theta \in \mathcal{E}^*(\mathbb{R}^{2d})$ such that $0 \leq \theta \leq 1$, $\theta = 0$ on $\mathbb{R}^{2d} \setminus \Omega_{r/4}$, $\theta = 1$ on $\Omega_{3r/4}$ and for every $h > 0$ there exists $C > 0$, resp. there exist $h > 0$ and $C > 0$, such that $|D_x^\beta D_y^\gamma \theta(x, y)| \leq Ch^{|\beta|+|\gamma|} M_{\beta+\gamma}$, for all $(x, y) \in \mathbb{R}^{2d}$, $\alpha, \beta \in \mathbb{N}^d$.*

Proof. Let $f(x, y) = 1$ on $\Omega_{r/2}$ and $f(x, y) = 0$ on $\mathbb{R}^{2d} \setminus \Omega_{r/2}$. Let $\mu \in \mathcal{D}^*(\mathbb{R}^{2d})$ is such that $\mu \geq 0$ with support in the closed ball with center at the origin and radius $r/16$ and $\int_{\mathbb{R}^{2d}} \mu(x, y) dx dy = 1$. Put $\theta = f * \mu$. Then, one easily checks that θ satisfies the conditions in the lemma. \square

Proposition 2.3. *Let $a \in \Pi_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{3d})$ and A be the operator corresponding to a as defined above. The kernel K of this operator is an element of $\mathcal{C}^\infty(\Omega_r)$ for every $0 < r < 1$ and for every such Ω_r and every $h > 0$, resp. there exists $h > 0$, such that*

$$\sup_{\beta, \gamma \in \mathbb{N}^d} \sup_{(x, y) \in \overline{\Omega_r}} \frac{h^{\beta+\gamma} |D_x^\beta D_y^\gamma K(x, y)| e^{M(h|(x, y)|)}}{M_{\beta+\gamma}} < \infty. \quad (8)$$

Moreover, if there exists r , $0 < r < 1$, such that $a(x, y, \xi) = 0$ for $(x, y, \xi) \in (\mathbb{R}^{2d} \setminus \Omega_r) \times \mathbb{R}^d$ then $K \in \mathcal{S}^*(\mathbb{R}^{2d})$, i.e. A is $*$ -regularizing.

Proof. Let $\psi_n \in \mathcal{D}^*(\mathbb{R}^d)$ be as in lemma 2.4, where R will be chosen later. Then, note that the sum $\sum_{n=0}^{\infty} 1_y \otimes \psi_n(\xi)$ converges to $1_{y, \xi}$ in $\mathcal{E}^*(\mathbb{R}_{y, \xi}^{2d})$ (with 1_y we denote the function of variable y that is identically equal to 1, similarly $1_{y, \xi}$ is the function of variables (y, ξ) that is identically equal to 1). Because $a(x, y, \xi)$ is an element of $\mathcal{E}^*(\mathbb{R}_{y, \xi}^{2d})$, for every fixed x , we get

$$a(x, y, \xi) = a(x, y, \xi) \sum_{n=0}^{\infty} \psi_n(\xi) = \sum_{n=0}^{\infty} (\psi_n(\xi) a(x, y, \xi)),$$

in $\mathcal{E}^*(\mathbb{R}_{y, \xi}^{2d})$. Let $u \in \mathcal{S}^*(\mathbb{R}^d)$. Because $1/P_l(y - x)$ and $1/P_l(\xi)$, resp. $1/P_{l_p}(y - x)$ and $1/P_{l_p}(\xi)$ are elements of $\mathcal{E}^*(\mathbb{R}_{y, \xi}^{2d})$, for $*$ = (M_p) , resp. $*$ = $\{M_p\}$, for fixed x , we get

$$\begin{aligned} & \frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y - x)} P_l(D_\xi) (a(x, y, \xi) u(y)) \right) \\ &= \sum_{n=0}^{\infty} \frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y - x)} P_l(D_\xi) (a(x, y, \xi) \psi_n(\xi) u(y)) \right), \end{aligned}$$

in the (M_p) case and with P_{l_p} in place of P_l in the $\{M_p\}$ case, in $\mathcal{E}^*(\mathbb{R}_{y, \xi}^{2d})$. If we choose l small enough such that $|P_l(\xi)| \geq c_1 e^{M(r|\xi|)} \geq c'_1 e^{2M(r'|\xi|)}$, where $r' > 0$ is such that $\int_{\mathbb{R}^d} e^{M(m|\xi|)} e^{-M(r'|\xi|)} d\xi < \infty$, by the properties of ψ_n similarly as in the proof of lemma 2.2, we obtain

$$\left| \frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y - x)} P_l(D_\xi) (a(x, y, \xi) \psi_n(\xi) u(y)) \right) \right| \leq C \frac{e^{M(m|x|)} e^{M(m|\xi|)}}{e^{M(r'|\xi|)} e^{M(r' R m_n)}} \cdot \frac{e^{M(m|y|)}}{e^{M(s|y|)}}$$

in the (M_p) case, where m is such that $a \in \Pi_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{3d}; m)$ and s is from the $\mathcal{S}^{(M_p)}$ - seminorms of u . Respectively, if we choose $(l_p) \in \mathfrak{R}$ small enough such that $|P_{l_p}(\xi)| \geq c'' e^{N_{r_p}(|\xi|)} \geq c_1'' e^{2N_{r_p'}(|\xi|)}$, where $(r_p') \in \mathfrak{R}$ is such that $\int_{\mathbb{R}^d} e^{N_{k_p}(|\xi|)} e^{-N_{r_p'}(|\xi|)} d\xi < \infty$, we get

$$\left| \frac{1}{P_{l_p}(\xi)} P_{l_p}(D_y) \left(\frac{1}{P_{l_p}(y-x)} P_{l_p}(D_\xi) (a(x, y, \xi) \psi_n(\xi) u(y)) \right) \right| \leq C \frac{e^{N_{k_p}(|x|)} e^{N_{k_p}(|\xi|)}}{e^{N_{r_p'}(|\xi|)} e^{N_{r_p'}(Rm_n)}} \cdot \frac{e^{N_{k_p}(|y|)}}{e^{M(s|y|)}}$$

in the $\{M_p\}$ case, where (k_p) is such that (6) holds for a and s depends on u . Hence, by dominated convergence,

$$\begin{aligned} Au(x) &= \frac{1}{(2\pi)^d} \sum_{n=0}^{\infty} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} \frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) (a(x, y, \xi) \psi_n(\xi) u(y)) \right) dy d\xi \\ &= \frac{1}{(2\pi)^d} \sum_{n=0}^{\infty} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} a(x, y, \xi) \psi_n(\xi) u(y) dy d\xi, \end{aligned}$$

in the (M_p) case, resp. the same but with P_{l_p} in place of P_l in the $\{M_p\}$ case and the convergence is uniform for x in compact subsets of \mathbb{R}^d and in $\mathcal{S}'^*(\mathbb{R}^d)$. For simpler notation, put $a_n(x, y, \xi) = a(x, y, \xi) \psi_n(\xi)$ and A_n for the associated operator to a_n . Then, we get

$$Au(x) = \sum_{n=0}^{\infty} A_n u(x), \text{ where the convergence is uniform for } x \text{ in compact subsets of } \mathbb{R}^d \text{ and}$$

in $\mathcal{S}'^*(\mathbb{R}^d)$. So $\sum_{k=0}^n A_k \rightarrow A$, when $n \rightarrow \infty$, in $\mathcal{L}_\sigma(\mathcal{S}^*(\mathbb{R}^d), \mathcal{S}'^*(\mathbb{R}^d))$. \mathcal{S}^* is barreled, so,

by the Banach - Steinhaus theorem (see [17], theorem 4.6), it follows that $\sum_{k=0}^n A_k \rightarrow A$,

when $n \rightarrow \infty$, in the topology of precompact convergence. But \mathcal{S}^* is Montel space, so the convergence holds in $\mathcal{L}_b(\mathcal{S}^*(\mathbb{R}^d), \mathcal{S}'^*(\mathbb{R}^d))$ (the topology of bounded convergence). Hence, if we denote by $K(x, y)$ the kernel of A and by K_n the kernel of A_n , by proposition

1.2, we get $K = \sum_{n=0}^{\infty} K_n$, where the convergence holds in $\mathcal{S}'^*(\mathbb{R}^{2d})$. Now, observe that

$$K_n(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} a(x, y, \xi) \psi_n(\xi) d\xi$$

and K_n is a \mathcal{C}^∞ function. Take R such that $Rm_1 \geq 1$. Later on we will impose more conditions on R . Let $r \in (0, 1)$ be fixed. First, we will observe the (M_p) case. There exists $m > 0$ such that $a \in \Pi_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{3d}; h, m)$, for all $h > 0$. Let m' be arbitrary but fixed positive real number. We want to prove (8) for this m' . Obviously, without losing generality, we can assume that $m' \geq 1$. Let $(x, y) \in \Omega_r$ be arbitrary but fixed. Let $q \in \{1, \dots, d\}$ be such that $|x_q - y_q| \geq |x_j - y_j|$, for all $j \in \{1, \dots, d\}$. Then $|x_q - y_q| > \frac{r}{d} \langle x \rangle$.

We calculate

$$\begin{aligned}
& D_x^\beta D_y^\gamma K_n(x, y) \\
&= \frac{1}{(2\pi)^d} \sum_{\substack{\beta'+\beta''=\beta \\ \gamma'+\gamma''=\gamma}} \sum_{k=0}^n \sum_{\substack{k'+k''=k \\ k'' \leq \beta_q''+\gamma_q''}} \binom{\beta}{\beta'} \binom{\gamma}{\gamma'} \binom{n}{k} \binom{k}{k'} \frac{(\beta''+\gamma'')!}{(\beta''+\gamma''-e_q k'')!} \frac{(-1)^{|\gamma''|+n}}{(x_q-y_q)^n i^{k''}} \\
&\quad \cdot \int_{\mathbb{R}^d} e^{i(x-y)\xi} \frac{1}{P_l(y-x)} P_l(D_\xi) \left(\xi^{\beta''+\gamma''-e_q k''} D_{\xi_q}^{k'} D_x^{\beta'} D_y^{\gamma'} a(x, y, \xi) D_{\xi_q}^{n-k} \psi_n(\xi) \right) d\xi.
\end{aligned}$$

On Ω_r we have the following inequality

$$|(x, y)| \leq |x| + |y| \leq \langle x \rangle + |x - y| + |x| \leq 2\langle x \rangle + |x - y| \leq \left(\frac{2}{r} + 1 \right) |x - y|. \quad (9)$$

Hence, by using proposition 3.6 of [9], we can find $m'' > 0$ such that $e^{M(m''|x-y|)} \geq c'' e^{M(m|x|)} e^{M(m|y|)} e^{M(m'|(x,y)|)}$ on Ω_r . Take $l' \geq m''$. Then we have

$$e^{M(l'|\xi|)} \geq c''' e^{M(m''|\xi|)}. \quad (10)$$

By proposition 1.1, we can find small enough $l > 0$ such that $|P_l(\xi)| \geq c'' e^{M(l'|\xi|)}$. On the other, hand if we represent $P_l(D)$ as $\sum_\alpha c_\alpha D^\alpha$, then there exist $C'_1 > 0$ and $L_0 > 0$ such that $|c_\alpha| \leq C'_1 L_0^{|\alpha|} / M_\alpha$. We will estimate the part in the integral for $n \in \mathbb{Z}_+$ as follows

$$\begin{aligned}
& \left| \frac{1}{P_l(y-x)} P_l(D_\xi) \left(\xi^{\beta''+\gamma''-e_q k''} D_{\xi_q}^{k'} D_x^{\beta'} D_y^{\gamma'} a(x, y, \xi) D_{\xi_q}^{n-k} \psi_n(\xi) \right) \right| \\
& \leq \frac{1}{|P_l(y-x)|} \sum_\alpha |c_\alpha| \sum_{\alpha' \leq \alpha} \sum_{\substack{\alpha''+\alpha'''=\alpha' \\ \alpha''' \leq \beta''+\gamma''-e_q k''}} \binom{\alpha}{\alpha'} \binom{\alpha'}{\alpha''} \frac{(\beta''+\gamma''-e_q k'')!}{(\beta''+\gamma''-e_q k''-\alpha''')!} \\
& \quad \cdot |\xi|^{\beta''+\gamma''-e_q k''-|\alpha'''|} \left| D_\xi^{\alpha''+e_q k'} D_x^{\beta'} D_y^{\gamma'} a(x, y, \xi) \right| \left| D_\xi^{\alpha-\alpha'+e_q(n-k)} \psi_n(\xi) \right| \\
& \leq C_1 e^{-M(l'|x-y|)} \sum_\alpha |c_\alpha| \sum_{\alpha' \leq \alpha} \sum_{\substack{\alpha''+\alpha'''=\alpha' \\ \alpha''' \leq \beta''+\gamma''-e_q k''}} \binom{\alpha}{\alpha'} \binom{\alpha'}{\alpha''} \frac{(\beta''+\gamma''-e_q k'')!}{(\beta''+\gamma''-e_q k''-\alpha''')!} \\
& \quad \cdot |\xi|^{\beta''+\gamma''-e_q k''-|\alpha'''|} \cdot \frac{h_1^{|\alpha|-|\alpha'|+n-k} M_{\alpha-\alpha'+n-k}}{(Rm_n)^{|\alpha|-|\alpha'|+n-k}} \\
& \quad \cdot \frac{h^{|\alpha''|+|\beta'|+|\gamma'|+k'} \langle x-y \rangle^{\rho|\alpha''|+\rho k'+\rho|\beta'|+\rho|\gamma'|} A_{\alpha''+k'} B_{\beta'+\gamma'} e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m|y|)}}{\langle (x, y, \xi) \rangle^{\rho|\alpha''|+\rho k'+\rho|\beta'|+\rho|\gamma'|}},
\end{aligned}$$

on the support of ψ_n . Note that $\langle x-y \rangle \leq 2(1+|x|^2+|y|^2)^{1/2} \leq 2\langle (x, y, \xi) \rangle$. Hence

$$\langle x-y \rangle^{\rho|\alpha''|+\rho|\beta'|+\rho|\gamma'|} \leq 2^{\rho|\alpha''|+\rho|\beta'|+\rho|\gamma'|} \langle (x, y, \xi) \rangle^{\rho|\alpha''|+\rho|\beta'|+\rho|\gamma'|}.$$

Also, $(\beta''+\gamma''-e_q k'')! \leq 2^{|\beta''+\gamma''-e_q k''|} (\beta''+\gamma''-e_q k''-\alpha''')! \alpha'''!$. Moreover $B_{\beta'+\gamma'} \leq c'_0 L^{|\beta'|+|\gamma'|} M_{\beta'+\gamma'}$ and $A_{\alpha''+k'} \leq c'_0 L^{|\alpha''|+k'} M_{\alpha''+k'}^\rho$. Let $T_n = \{\xi \in \mathbb{R}^d | 2Rm_n \leq \langle \xi \rangle \leq 3Rm_{n+1}\}$.

By construction, $\text{supp } \psi_n \subseteq T_n$. Note that, on T_n

$$\frac{|\xi|^{|\beta''+\gamma''-e_q k''|-|\alpha'''|}}{\langle(x, y, \xi)\rangle^{\rho k'}} \leq \frac{\langle\xi\rangle^{|\beta''+\gamma''-e_q k''|-|\alpha'''|}}{\langle\xi\rangle^{\rho k'}} \leq \frac{(3Rm_{n+1})^{|\beta''|+|\gamma''|}}{(3Rm_{n+1})^{|\alpha'''|+k''} (2Rm_n)^{\rho k'}}.$$

Because m_n is monotonically increasing, $m_n^{n-k} \geq m_n \cdot m_{n-1} \cdots m_{k+1} = M_n/M_k \geq M_{n-k}$ and similarly, $m_n^{k'} \geq M_{k'}$ and $m_n^{k''} \geq M_{k''}$. Moreover, there exists $\tilde{c} > 0$ such that $M_p^\rho \leq \tilde{c}M_p$. We use this to estimate the above integral. By Fatou's lemma we have $\int_{\mathbb{R}^d} |\sum \dots| d\xi \leq \sum \int_{\mathbb{R}^d} |\dots| d\xi$. Considering the parts that are depended on $\alpha, \alpha', \alpha''$ and α''' , after using the above inequalities, one obtains

$$\begin{aligned} & e^{M(3m_{n+1})} \sum_{\alpha} \sum_{\alpha' \leq \alpha} \sum_{\substack{\alpha''+\alpha'''=\alpha' \\ \alpha''' \leq \beta''+\gamma''-e_q k''}} \binom{\alpha}{\alpha'} \binom{\alpha'}{\alpha''} \frac{L_0^{|\alpha|-|\alpha'''|} L_0^{|\alpha'''|}}{M_{\alpha}} \\ & \quad \cdot \frac{\alpha'''!(2h)^{|\alpha''|} L^{|\alpha''|+k'} M_{\alpha''+k'}^\rho h_1^{|\alpha|-|\alpha'|} M_{\alpha-\alpha'+n-k}}{(3Rm_{n+1})^{|\alpha'''|+k''} (2Rm_n)^{\rho k'} (Rm_n)^{|\alpha|-|\alpha'|+n-k}} \cdot |T_n| \\ & \leq C_2 e^{M(3m_{n+1})} \sum_{\alpha} \sum_{\alpha' \leq \alpha} \sum_{\substack{\alpha''+\alpha'''=\alpha' \\ \alpha''' \leq \beta''+\gamma''-e_q k''}} \binom{\alpha}{\alpha'} \binom{\alpha'}{\alpha''} \frac{L_0^{|\alpha|-|\alpha'''|}}{M_{\alpha}} \\ & \quad \cdot \frac{(2h)^{|\alpha''|} L^{|\alpha|+n} h_1^{|\alpha|-|\alpha'|} H^{|\alpha|+n} M_{\alpha'''} M_{\alpha''} M_{k'}^\rho M_{\alpha-\alpha'} M_{n-k}}{(Rm_n)^{|\alpha'''|+k''} (Rm_n)^{\rho k'} (Rm_n)^{|\alpha|-|\alpha'|+n-k}} \cdot |T_n| \\ & \leq \frac{C_2 |T_n| e^{M(3m_{n+1})}}{R^{\rho n}} \sum_{\alpha} \sum_{\alpha' \leq \alpha} \sum_{\substack{\alpha''+\alpha'''=\alpha' \\ \alpha''' \leq \beta''+\gamma''-e_q k''}} \binom{\alpha}{\alpha'} \binom{\alpha'}{\alpha''} \frac{(HL)^{|\alpha|+n} (2L_0 h)^{|\alpha''|} (L_0 h_1)^{|\alpha|-|\alpha'|}}{(RM_1)^{|\alpha|-|\alpha''|} m_n^{k''}} \\ & \leq \frac{C_2 |T_n| (HL)^n e^{M(3m_{n+1})}}{R^{\rho n} M_{k''}} \sum_{\alpha} \frac{(HL)^{|\alpha|}}{(RM_1)^{|\alpha|}} \sum_{\alpha' \leq \alpha} \binom{\alpha}{\alpha'} (1+2L_0 h R M_1)^{|\alpha'|} (L_0 h_1)^{|\alpha|-|\alpha'|} \\ & = \frac{C_2 |T_n| (HL)^n e^{M(3m_{n+1})}}{R^{\rho n} M_{k''}} \sum_{\alpha} \left(\frac{HL}{RM_1} + 2h H L L_0 + \frac{h_1 H L L_0}{RM_1} \right)^{|\alpha|}. \end{aligned}$$

Take R such that $\frac{HL}{RM_1} + \frac{1}{8} + \frac{1}{RM_1} \leq \frac{1}{2}$ and take h and h_1 small enough such that $2h H L L_0 \leq 1/8$ and $h_1 H L L_0 \leq 1$. Then, the sum will be uniformly convergent for all h and h_1 for which the previous inequalities hold. The choice of R depends only on A_p, B_p and M_p (and not on L_0 , hence not on the operator P_l). Also, the choice of h and h_1 depend on A_p, B_p, M_p and the operator P_l , but not on R . Before we continue, note that, from the way we choose q , we have the following inequality

$$1 + |x - y|^2 \leq \langle x \rangle^2 + d|x_q - y_q|^2 \leq \frac{d^2}{r^2} |x_q - y_q|^2 + d|x_q - y_q|^2 \leq \left(\frac{d}{r} + d \right)^2 |x_q - y_q|^2.$$

For shorter notation, put $r_1 = \frac{d}{r} + d$. So, we obtain $\langle x - y \rangle \leq r_1 |x_q - y_q|$. Now, for the estimate of $|D_x^\beta D_y^\gamma K_n(x, y)|$, by using (10), we obtain

$$\begin{aligned}
& |D_x^\beta D_y^\gamma K_n(x, y)| \\
& \leq C_3 \sum_{\substack{\beta' + \beta'' = \beta \\ \gamma' + \gamma'' = \gamma}} \sum_{k=0}^n \sum_{\substack{k' + k'' = k \\ k'' \leq \beta_q'' + \gamma_q''}} \binom{\beta}{\beta'} \binom{\gamma}{\gamma'} \binom{n}{k} \binom{k}{k'} \frac{(\beta'' + \gamma'')!}{(\beta'' + \gamma'' - e_q k'')!} \frac{\langle x - y \rangle^{\rho k'}}{|x_q - y_q|^n} \\
& \quad \cdot 2^{|\beta'' + \gamma'' - e_q k''|} (3Rm_{n+1})^{|\beta''| + |\gamma''|} h^{|\beta'| + |\gamma'| + k'} 2^{|\beta'| + |\gamma'|} L^{|\beta'| + |\gamma'|} M_{\beta' + \gamma'} h_1^{n-k} \\
& \quad \cdot e^{-M(l'|y-x|)} e^{M(m|x|)} e^{M(m|y|)} \frac{|T_n| (HL)^n e^{M(3mRm_{n+1})}}{R^{\rho n} M_{k''}} \\
& \leq C_3 r_1^n \sum_{\substack{\beta' + \beta'' = \beta \\ \gamma' + \gamma'' = \gamma}} \sum_{k=0}^n \sum_{k' + k'' = k} \binom{\beta}{\beta'} \binom{\gamma}{\gamma'} \binom{n}{k} \binom{k}{k'} \frac{4^{|\beta''| + |\gamma''|} k''!}{M_{k''}} \cdot \frac{h_1^{k''}}{h_1^{k''}} \\
& \quad \cdot \frac{1}{2^{k''} (m'R)^{|\beta''| + |\gamma''|}} \frac{(3m'R^2 m_{n+1})^{|\beta''| + |\gamma''|}}{M_{\beta'' + \gamma''}} h^{|\beta'| + |\gamma'| + k'} (2L)^{|\beta'| + |\gamma'|} h_1^{n-k} M_{\beta + \gamma} \\
& \quad \cdot e^{-M(m'|(x,y)|)} \frac{|T_n| (HL)^n e^{M(3mRm_{n+1})}}{R^{\rho n}}.
\end{aligned}$$

Note that $\frac{(3m'R^2 m_{n+1})^{|\beta''| + |\gamma''|}}{M_{\beta'' + \gamma''}} \leq e^{M(3m'R^2 m_{n+1})}$. Also, by using (M.2), we obtain

$$|T_n| = \omega_d \left((9R^2 m_{n+1}^2 - 1)^{d/2} - (4R^2 m_n^2 - 1)^{d/2} \right) \leq \omega_d (3Rm_{n+1})^d \leq \omega_d (3c_0 R M_1)^d H^{(n+1)d},$$

where ω_d is the volume of the d -dimensional unit ball. By proposition 3.6 of [9]

$$e^{M(3mRm_{n+1})} e^{M(3m'R^2 m_{n+1})} \leq c_0 e^{M(3Hm'R^2 m_{n+1})},$$

where we take $R \geq m$ (which depends only on a). We obtain

$$\begin{aligned}
& |D_x^\beta D_y^\gamma K_n(x, y)| \\
& \leq C_4 (3c_0 R M_1 H)^d \frac{M_{\beta + \gamma} (HL)^n H^{nd} e^{M(3Hm'R^2 m_{n+1})} r_1^n}{e^{M(m'|(x,y)|)} R^{\rho n}} \left(\frac{4}{m'R} + 2hL \right)^{|\beta| + |\gamma|} \left(h_1 + h + \frac{h_1}{2} \right)^n.
\end{aligned}$$

By lemma 2.3 we have

$$e^{M(3Hm'R^2 m_{n+1})} \leq c_0 H^{2(3c_0 H m' R^2 + 2)(n+1)} = c_0 H^{2(3c_0 H m' R^2 + 2)} \left(H^{2(3c_0 H m' R^2 + 2)} \right)^n.$$

Take $R^\rho > H^{d+1} L r_1$ and $R \geq 8$. For the fixed m' in the beginning of the proof, choose h small enough such that $2hL \leq 1/(2m')$. Then $\frac{4}{m'R} + 2hL \leq \frac{1}{m'}$. For the chosen R , choose h and h_1 smaller then the chosen before such that $H^{2(3c_0 H m' R^2 + 2)} \left(h_1 + h + \frac{h_1}{2} \right) \leq 1$. (Note that the choice of R and hence the choice of ψ_n , $n \in \mathbb{N}$, depends only on A_p , B_p , M_p

and a , but not on the operator P_l or m' .) Then $\sum_{n=1}^{\infty} |D_x^\beta D_y^\gamma K_n(x, y)|$ will converge and we have the following estimate

$$\sum_{n=1}^{\infty} |D_x^\beta D_y^\gamma K_n(x, y)| \leq C \frac{M_{\beta+\gamma}}{e^{M(m'|(x,y)|)} m'^{|\beta|+|\gamma|}}.$$

For $|D_x^\beta D_y^\gamma K_0(x, y)|$, by similar procedure, we obtain the same estimate. Hence (8) holds and the proof for the (M_p) case is complete.

The $\{M_p\}$ case. We will prove that for every $(t_p), (t'_p) \in \mathfrak{R}$,

$$\sup_{\beta, \gamma \in \mathbb{N}^d} \sup_{(x, y) \in \overline{\Omega_r}} \frac{|D_x^\beta D_y^\gamma K(x, y)| e^{N_{t_p}(|(x, y)|)}}{T'_{\beta+\gamma} M_{\beta+\gamma}} < \infty, \quad (11)$$

for every fixed $0 < r < 1$, where $T'_{\beta+\gamma} = \prod_{j=1}^{|\beta|+|\gamma|} t'_j$ and $T'_0 = 1$. From this, the claim in the lemma follows. To prove this, fix $0 < r < 1$ and take $\theta \in \mathcal{E}^{\{M_p\}}(\mathbb{R}^{2d})$ as in lemma 2.5. Define $\tilde{K} = K\theta$. Then \tilde{K} is \mathcal{C}^∞ function and for every $(t_p), (t'_p) \in \mathfrak{R}$,

$\sup_{\beta, \gamma \in \mathbb{N}^d} \sup_{(x, y) \in \mathbb{R}^{2d}} \frac{|D_x^\beta D_y^\gamma \tilde{K}(x, y)| e^{N_{t_p}(|(x, y)|)}}{T'_{\beta+\gamma} M_{\beta+\gamma}} < \infty$. Hence $\tilde{K} \in \mathcal{S}^{\{M_p\}}(\mathbb{R}^{2d})$. So, there exists $h > 0$ such that $\sup_{\beta, \gamma \in \mathbb{N}^d} \sup_{(x, y) \in \mathbb{R}^{2d}} \frac{h^{|\beta|+|\gamma|} |D_x^\beta D_y^\gamma \tilde{K}(x, y)| e^{M(h|(x, y)|)}}{M_{\beta+\gamma}} < \infty$. But, $\tilde{K}(x, y) = K(x, y)$ on $\Omega_{3r/4}$ and the desired estimate follows. Now, to prove (11). Let $a \in \Pi_{A_p, B_p}^{\{M_p\}, \infty}(\mathbb{R}^{3d})$. Then there exists $h > 0$ such that $a \in \Pi_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{3d}; h, m)$, for all $m > 0$. By lemma 2.1, there exist $(k_p) \in \mathfrak{R}$ and $c'_0 > 0$ such that

$$|D_\xi^\alpha D_x^\beta D_y^\gamma a(x, y, \xi)| \leq c'_0 \frac{h^{|\alpha|+|\beta|+|\gamma|} \langle x - y \rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|} A_\alpha B_{\beta+\gamma} e^{N_{k_p}(|\xi|)} e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)}}{\langle (x, y, \xi) \rangle^{\rho|\alpha|+\rho|\beta|+\rho|\gamma|}},$$

for all $\alpha, \beta, \gamma \in \mathbb{N}^d$ and $(x, y, \xi) \in \mathbb{R}^{3d}$. Let $(t_p), (t'_p) \in \mathfrak{R}$ be fixed. For $(l_p) \in \mathfrak{R}$ consider $P_{l_p}(\xi)$. By proposition 1.1, we can choose $P_{l_p}(\xi)$ such that, $|P_{l_p}(\xi)| \geq c'' e^{N_{l'_p}(|\xi|)}$ where $(l'_p) \in \mathfrak{R}$ is such that $e^{N_{l'_p}(|x-y|)} \geq c_1 e^{N_{k_p}(|x|)} e^{N_{k_p}(|y|)} e^{N_{t_p}(|(x, y)|)}$ on Ω_r . This is possible because of (9). On the other hand, if we represent $P_{l_p}(\xi) = \sum_{\alpha} c_\alpha \xi^\alpha$ then for every $L' > 0$

there exists $C' > 0$ such that $|c_\alpha| \leq C' L'^{|\alpha|} / M_\alpha$. By the same calculations, one obtains the same form for $D_x^\beta D_y^\gamma K_n(x, y)$ as in the (M_p) case, but with P_{l_p} in place of P_l . The prove continues in the same way as above. We will point out only the notable differences. The first difference is in the estimate of the part that is depended on $\alpha, \alpha', \alpha''$ and α''' (for $n \in \mathbb{Z}_+$) and the integral over \mathbb{R}_ξ^d , where in the $\{M_p\}$ case one obtains the estimate

$$\frac{C_2 |T_n| (HL)^n e^{N_{k_p}(3Rm_{n+1})}}{R^{\rho n} M_{k''}} \sum_{\alpha} \left(\frac{HL}{RM_1} + 2hHLL' + \frac{h_1 HLL'}{RM_1} \right)^{|\alpha|}.$$

The convergence of this sum follows from the fact that we can take R arbitrary large and L' arbitrary small. Moving on to the estimate of $|D_x^\beta D_y^\gamma K_n(x, y)|$, in similar fashion, one obtains the following

$$\begin{aligned} & |D_x^\beta D_y^\gamma K_n(x, y)| \\ & \leq C_3 r_1^n \sum_{\substack{\beta' + \beta'' = \beta \\ \gamma' + \gamma'' = \gamma}} \sum_{k=0}^n \sum_{k' + k'' = k} \binom{\beta}{\beta'} \binom{\gamma}{\gamma'} \binom{n}{k} \binom{k}{k'} \frac{12^{|\beta''| + |\gamma''|} k''! R^{|\beta''| + |\gamma''|}}{M_{k''}} \frac{1}{2^{k''}} \\ & \quad \cdot \frac{m_{n+1}^{|\beta''| + |\gamma''|}}{M_{\beta'' + \gamma''}} h^{|\beta'| + |\gamma'| + k'} (2L)^{|\beta'| + |\gamma'|} h_1^{n-k} M_{\beta + \gamma} e^{-N_{t_p}(|(x, y)|)} \frac{|T_n| (HL)^n e^{N_{k_p}(3Rm_{n+1})}}{R^{\rho n}}. \end{aligned}$$

By using the increasingness of m_p and (M.2), we obtain

$$\frac{m_{n+1}^{|\beta''| + |\gamma''|}}{M_{\beta'' + \gamma''}} \leq \frac{m_{n+2} \cdot m_{n+3} \cdot \dots \cdot m_{n+1+|\beta''| + |\gamma''|}}{M_{\beta'' + \gamma''}} = \frac{M_{n+1+\beta'' + \gamma''}}{M_{\beta'' + \gamma''} M_{n+1}} \leq c_0 H^{n+1+|\beta''| + |\gamma''|}.$$

We get the estimate:

$$|D_x^\beta D_y^\gamma K_n(x, y)| \leq C_4 \frac{M_{\beta + \gamma} |T_n| (H^2 L)^n e^{N_{k_p}(3Rm_{n+1})} r_1^n}{e^{N_{t_p}(|(x, y)|)} R^{\rho n}} (12RH + 2hL)^{|\beta| + |\gamma|} \left(h_1 + h + \frac{1}{2} \right)^n.$$

By lemma 2.3 $e^{N_{k_p}(3Rm_{n+1})} \leq cH^{n+1}$, where c depends only on (k_p) , R and M_p (does not depend on n). Now, if we use the same estimate for $|T_n|$ as in the (M_p) case, if we take large enough R , the sum $\sum_{n=1}^{\infty} |D_x^\beta D_y^\gamma K_n(x, y)|$ will converge and we obtain

$$\sum_{n=1}^{\infty} |D_x^\beta D_y^\gamma K(x, y)| \leq C \frac{M_{\beta + \gamma}}{e^{N_{t_p}(|(x, y)|)}} (12RH + 2hL)^{|\beta| + |\gamma|}.$$

One obtains similar estimates for $|D_x^\beta D_y^\gamma K_0(x, y)|$. Hence we obtain (11) and the proof for the $\{M_p\}$ case is complete. It remains to prove the fact that if there exists r , $0 < r < 1$, such that $a(x, y, \xi) = 0$ for $(x, y, \xi) \in (\mathbb{R}^{2d} \setminus \Omega_r) \times \mathbb{R}^d$ then $K \in \mathcal{S}^*(\mathbb{R}^{2d})$. But this trivially follows from the proved growth condition of $D_x^\beta D_y^\gamma K(x, y)$ and the fact that for $(x, y) \in \mathbb{R}^{2d} \setminus \Omega_r$, $K_n(x, y) = 0$ for all $n \in \mathbb{N}$, hence, $K = 0$ on $\mathbb{R}^{2d} \setminus \Omega_r$. \square

3 Symbolic calculus

Let $\rho_1 = \inf\{\rho \in \mathbb{R}_+ | A_p \subset M_p^\rho\}$ and $\rho_2 = \inf\{\rho \in \mathbb{R}_+ | B_p \subset M_p^\rho\}$ and put $\rho_0 = \max\{\rho_1, \rho_2\}$. Then $0 < \rho_0 \leq 1$ and for every ρ such that $\rho_0 \leq \rho \leq 1$, if the larger infimum can be reached, or, otherwise $\rho_0 < \rho \leq 1$, $A_p \subset M_p^\rho$ and $B_p \subset M_p^\rho$. So, for every such ρ , there exists $c'_0 > 0$ and $L > 0$ (which depend on ρ) such that, $A_p \leq c'_0 L^p M_p^\rho$, $B_p \leq c'_0 L^p M_p^\rho$. Moreover, because M_p tends to infinity, there exists $\tilde{c} > 0$ such that $M_p^\rho \leq \tilde{c} M_p$, for all such ρ . From

now on we suppose that $\rho_0 \leq \rho \leq 1$, if the larger infimum can be reached, or otherwise $\rho_0 < \rho \leq 1$.

For $t > 0$, put $Q_t = \{(x, \xi) \in \mathbb{R}^{2d} | \langle x \rangle < t, \langle \xi \rangle < t\}$ and $Q_t^c = \mathbb{R}^{2d} \setminus Q_t$. Denote by $FS_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; B, h, m)$ the vector space of all formal series $\sum_{j=0}^{\infty} a_j(x, \xi)$ such that $a_j \in \mathcal{C}^\infty(\text{int } Q_{Bm_j}^c)$, $D_\xi^\alpha D_x^\beta a_j(x, \xi)$ can be extended to continuous function on $Q_{Bm_j}^c$ for all $\alpha, \beta \in \mathbb{N}^d$ and

$$\sup_{j \in \mathbb{N}} \sup_{\alpha, \beta} \sup_{(x, \xi) \in Q_{Bm_j}^c} \frac{|D_\xi^\alpha D_x^\beta a_j(x, \xi)| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + 2j\rho} e^{-M(m|\xi|)} e^{-M(m|x|)}}{h^{|\alpha| + |\beta| + 2j} A_\alpha B_\beta A_j B_j} < \infty.$$

In the above, we use the convention $m_0 = 0$ and hence $Q_{Bm_0}^c = \mathbb{R}^{2d}$. It is easy to check that $FS_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; B, h, m)$ is a Banach space. Define

$$\begin{aligned} FS_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; B, m) &= \lim_{h \rightarrow 0} FS_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; B, h, m), \\ FS_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}) &= \lim_{B, m \rightarrow \infty} FS_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; B, m), \\ FS_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; B, h) &= \lim_{m \rightarrow 0} FS_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; B, h), \\ FS_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}) &= \lim_{B, h \rightarrow \infty} FS_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; B, h). \end{aligned}$$

Then, $FS_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; B, m)$ and $FS_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; B, h)$ are (F) -spaces. The inclusions $FS_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; B, m) \longrightarrow \prod_{j=0}^{\infty} \mathcal{E}^{(M_p)}(\text{int } Q_{Bm_j}^c)$ and $FS_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; B, h) \longrightarrow \prod_{j=0}^{\infty} \mathcal{E}^{\{M_p\}}(\text{int } Q_{Bm_j}^c)$, $\sum_{j=0}^{\infty} a_j \mapsto (a_0, a_1, a_2, \dots)$, are continuous, so $FS_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d})$ and $FS_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d})$ are Hausdorff l.c.s. Moreover, as inductive limits of barreled and bornological spaces they are barreled and bornological. Note, also, that the inclusions $\Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d}) \longrightarrow FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$, defined as $a \mapsto \sum_{j \in \mathbb{N}} a_j$, where $a_0 = a$ and $a_j = 0$, $j \geq 1$, is continuous.

Definition 3.1. Two sums, $\sum_{j \in \mathbb{N}} a_j, \sum_{j \in \mathbb{N}} b_j \in FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$, are said to be equivalent, in notation $\sum_{j \in \mathbb{N}} a_j \sim \sum_{j \in \mathbb{N}} b_j$, if there exist $m > 0$ and $B > 0$, resp. there exist $h > 0$ and $B > 0$, such that for every $h > 0$, resp. for every $m > 0$,

$$\sup_{N \in \mathbb{Z}_+} \sup_{\alpha, \beta} \sup_{(x, \xi) \in Q_{Bm_N}^c} \frac{|D_\xi^\alpha D_x^\beta \sum_{j < N} (a_j(x, \xi) - b_j(x, \xi))| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + 2N\rho}}{h^{|\alpha| + |\beta| + 2N} A_\alpha B_\beta A_N B_N}.$$

$$e^{-M(m|\xi|)}e^{-M(m|x|)} < \infty.$$

From now on, we assume that A_p and B_p satisfy (M.2). Without losing generality we can assume that the constants c_0 and H from the condition (M.2) for A_p and B_p are the same as the corresponding constants for M_p .

Theorem 3.1. *Let $a \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ be such that $a \sim 0$. Then, for every $\tau \in \mathbb{R}$, $\text{Op}_\tau(a)$ is $*$ -regularizing.*

Proof. First we will prove the following lemma.

Lemma 3.1. *Let $0 < l \leq 1$ and $B > 1$. There exists $C > 0$ depending on B , l and M_p and $\tilde{m} > 0$ depending only on B and M_p and not on l such that*

$$\inf \left\{ \frac{M_n}{l^n \rho^n} \mid n \in \mathbb{Z}_+, \rho \geq Bm_n \right\} \leq Ce^{-M(l\tilde{m}\rho)}, \text{ for all } \rho \geq BM_1.$$

Proof. For shorter notation put

$$f(\rho) = \inf \left\{ \frac{M_n}{l^n \rho^n} \mid n \in \mathbb{Z}_+, \rho \geq Bm_n \right\}$$

and $T_{\rho,0} = \{n \in \mathbb{Z}_+ \mid \rho \geq Bm_n\}$, $T_{\rho,1} = \{n \in \mathbb{Z}_+ \mid \rho < Bm_n\}$. Obviously $T_{\rho,0} \cup T_{\rho,1} = \mathbb{Z}_+$ and they are not empty. For $n \in \mathbb{Z}_+$, denote by $\mathbb{Z}_{+,n}$ the set $\{1, \dots, n\}$. By the properties of m_n , there exists $k \in \mathbb{Z}_+$ (which depends on ρ) such that $T_{\rho,0} = \{1, 2, \dots, k\}$. In the proof of lemma 2.3, we proved that, for $s \in \mathbb{Z}_+$, $\frac{m_{k+s+1}}{m_{k+1}} \geq \frac{s}{c_0(k+1)}$. Take $s = 2k([c_0] + 1)$, and for shorter notation, put $t = 2[c_0] + 2$. Then $m_{k+kt+1} > m_{k+1}$. For $q \in \mathbb{Z}_+$, we get $Bm_{k+kt+q} \geq Bm_{k+kt+1} > Bm_{k+1} \geq l\rho$. Then, for $q \in \mathbb{Z}_+$, we have

$$\frac{B^{k+kt+q}M_{k+kt+q}}{l^{k+kt+q}\rho^{k+kt+q}} = \frac{B^{k+kt+q-1}M_{k+kt+q-1}}{l^{k+kt+q-1}\rho^{k+kt+q-1}} \cdot \frac{Bm_{k+kt+q}}{l\rho} > \frac{B^{k+kt+q-1}M_{k+kt+q-1}}{l^{k+kt+q-1}\rho^{k+kt+q-1}}.$$

So, we obtain

$$e^{-M(l\rho/B)} = \inf_{n \in \mathbb{N}} \frac{B^n M_n}{l^n \rho^n} = \inf_{n \in \mathbb{Z}_+, k+kt} \frac{B^n M_n}{l^n \rho^n}, \quad (12)$$

for $\rho > BM_1/l$ (the infimum can not be obtained for $n = 0$). Now, let $0 \leq q \leq t$, $q \in \mathbb{N}$ and $n \in T_{\rho,0}$. One has

$$\frac{B^{n+qk}M_{n+qk}}{l^{n+qk}\rho^{n+qk}} \geq \frac{B^n M_n}{l^n \rho^n} \left(\frac{B^k M_k}{l^k \rho^k} \right)^q \geq f(\rho)^{q+1} \geq f(\rho)^{t+1},$$

where the last inequality holds because $f(\rho) \leq 1$ when $\rho > BM_1/l$. Hence, by (12), $e^{-M(l\rho/B)} \geq f(\rho)^{t+1}$, for $\rho > BM_1/l$. Repeated use of proposition 3.6 of [9] yields

$$(t+1)M\left(\frac{l\rho}{BH^{t+1}}\right) \leq 2^{t+1}M\left(\frac{l\rho}{BH^{t+1}}\right) \leq M\left(\frac{l\rho}{B}\right) + \ln c',$$

i.e. $f(\rho) \leq e^{-\frac{1}{t+1}M(l\rho/B)} \leq Ce^{-M(l\tilde{m}\rho)}$, $\forall \rho > BM_1/l$, where we put $\tilde{m} = 1/(BH^{t+1})$, which depends only on B and the sequence M_p (recall that $t = 2[c_0] + 2$). For $BM_1 \leq \rho \leq BM_1/l$, $f(\rho)$ is bounded so the same inequality holds, possibly with another C . \square

We continue the proof of the theorem. It is enough to prove that $a \in \mathcal{S}^*$, because then the claim will follow from proposition 2.1. Because $a \sim 0$, in the (M_p) case, there exist $m > 0$ and $B > 0$, such that for every $h > 0$ there exists $C > 0$, resp. in the $\{M_p\}$ case, there exist $h > 0$ and $B > 0$, such that for every $m > 0$ there exists $C > 0$, such that

$$\begin{aligned} |D_\xi^\alpha D_x^\beta a(x, \xi)| &\leq C \frac{h^{|\alpha|+|\beta|+2N} A_\alpha B_\beta A_N B_N e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x, \xi) \rangle^{\rho|\alpha|+\rho N} \langle (x, \xi) \rangle^{\rho|\beta|+\rho N}} \\ &\leq C_1 \frac{h^{|\alpha|+|\beta|} A_\alpha B_\beta e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x, \xi) \rangle^{\rho|\alpha|+\rho|\beta|}} \cdot \frac{(hL)^{2N} M_N^{2\rho}}{\langle (x, \xi) \rangle^{2N\rho}}, \end{aligned}$$

for all $N \in \mathbb{Z}_+$, $\alpha, \beta \in \mathbb{N}^d$, $(x, \xi) \in Q_{Bm_N}^c$. It is obvious that without losing generality we can assume that $B > 1$. In the (M_p) case let $m' > 0$ be arbitrary but fixed. Let $(x, \xi) \in Q_{Bm_1}^c$. Then, there exists $N \in \mathbb{Z}_+$ such that $(x, \xi) \in Q_{Bm_{N+1}} \setminus Q_{Bm_N}$. We estimate as follows

$$\begin{aligned} &\frac{m'^{|\alpha|+|\beta|} |D_\xi^\alpha D_x^\beta a(x, \xi)| e^{M(m'|(x, \xi)|)}}{M_{\alpha+\beta}} \\ &\leq C_1 \frac{(m'h)^{|\alpha|+|\beta|} A_\alpha B_\beta e^{M(m|\xi|)} e^{M(m|x|)} e^{M(m'|(x, \xi)|)}}{\langle (x, \xi) \rangle^{\rho|\alpha|+\rho|\beta|} M_{\alpha+\beta}} \cdot \frac{(hL)^{2N} M_N^{2\rho}}{\langle (x, \xi) \rangle^{2N\rho}} \\ &\leq C_2 \frac{(m'hL)^{|\alpha|+|\beta|} M_{\alpha+\beta}^\rho e^{2M(mBm_{N+1})} e^{M(2m'Bm_{N+1})}}{\langle (x, \xi) \rangle^{\rho|\alpha|+\rho|\beta|} M_{\alpha+\beta}} \cdot \frac{(hL)^{2N} M_N^{2\rho}}{(Bm_N)^{2N\rho}} \\ &\leq C_3 (m'hL)^{|\alpha|+|\beta|} (hL)^{2N} e^{2M(mBm_{N+1})} e^{M(2m'Bm_{N+1})}, \end{aligned}$$

where, in the last inequality, we used $m_N^N \geq M_N$. By lemma 2.3, we have

$$\begin{aligned} e^{2M(mBm_{N+1})} e^{M(2m'Bm_{N+1})} &\leq c_0^3 H^{4(c_0mB+2)(N+1)} H^{2(2c_0m'B+2)(N+1)} \\ &= c_0^3 H^{4(c_0mB+2)} H^{2(2c_0m'B+2)} \left(H^{4(c_0mB+2)} H^{2(2c_0m'B+2)} \right)^N. \end{aligned}$$

Take h small enough such that $m'hL \leq 1$ and $h^2 L^2 H^{4(c_0mB+2)} H^{2(2c_0m'B+2)} \leq 1$. We get

$$\frac{m'^{|\alpha|+|\beta|} |D_\xi^\alpha D_x^\beta a(x, \xi)| e^{M(m'|(x, \xi)|)}}{M_{\alpha+\beta}} \leq C$$

for all $\alpha, \beta \in \mathbb{N}^d$ and $(x, \xi) \in Q_{Bm_1}^c$. For $(x, \xi) \in Q_{Bm_1}$ the same estimate will hold, possibly for another $C > 0$, because $a \in \Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}) \subseteq \mathcal{E}^{(M_p)}(\mathbb{R}^{2d})$ and Q_{Bm_1} is bounded.

In the $\{M_p\}$ case, by the above observations, we have

$$|D_\xi^\alpha D_x^\beta a(x, \xi)| \leq C_1 \frac{h^{|\alpha|+|\beta|} A_\alpha B_\beta e^{M(m|\xi|)} e^{M(m|x|)}}{\langle (x, \xi) \rangle^{\rho|\alpha|+\rho|\beta|}}$$

$$\cdot \left(\inf \left\{ \frac{(h^{1/\rho} L^{1/\rho})^N M_N}{\langle (x, \xi) \rangle^N} \middle| N \in \mathbb{Z}_+, (x, \xi) \in Q_{Bm_N}^c \right\} \right)^{2\rho}.$$

and it is obvious that without losing generality we can assume that $h \geq 1$ and $L \geq 1$ (L is the constant from $A_p \subset M_p^\rho$ and $B_p \subset M_p^\rho$). Now, note that

$$\begin{aligned} & \inf \left\{ \frac{(h^{1/\rho} L^{1/\rho})^N M_N}{\langle (x, \xi) \rangle^N} \middle| N \in \mathbb{Z}_+, (x, \xi) \in Q_{Bm_N}^c \right\} \\ & \leq \inf \left\{ \frac{(h^{1/\rho} L^{1/\rho})^N M_N}{\langle (x, \xi) \rangle^N} \middle| N \in \mathbb{Z}_+, \langle (x, \xi) \rangle \geq 2Bm_N \right\} \leq C' e^{-M(\tilde{m}\langle (x, \xi) \rangle)/(hL)^{1/\rho}}, \end{aligned}$$

for all $\langle (x, \xi) \rangle \geq 2BM_1$, where in the last inequality we use the above lemma with $l = (hL)^{-1/\rho} \leq 1$. By proposition 3.6 of [9], $e^{M(m|\xi|)} e^{M(m|x|)} \leq c_0 e^{M(mH|(x, \xi)|)}$. Using the fact that $A_p \subset M_p^\rho$ and $B_p \subset M_p^\rho$ and the above inequalities, we get

$$|D_\xi^\alpha D_x^\beta a(x, \xi)| \leq C_2 (L^2 h)^{|\alpha|+|\beta|} M_{\alpha+\beta} e^{M(mH|(x, \xi)|)} e^{-M(|(x, \xi)|\tilde{m}/(hL)^{1/\rho})},$$

for all $\alpha, \beta \in \mathbb{N}^d$ and $\langle (x, \xi) \rangle \geq 2BM_1$. For $\langle (x, \xi) \rangle \leq 2BM_1$ the same estimate will hold, possibly for another $C > 0$ and $\tilde{h} > 0$ instead of $L^2 h$, because $a \in \Gamma_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}) \subseteq \mathcal{E}^{\{M_p\}}(\mathbb{R}^{2d})$ and the set $\{(x, \xi) \in \mathbb{R}^{2d} | \langle (x, \xi) \rangle \leq 2BM_1\}$ is bounded. m can be arbitrary small, so if we take m small enough we have $e^{M(mH|(x, \xi)|)} e^{-M(|(x, \xi)|\tilde{m}/(hL)^{1/\rho})} \leq C_3 e^{-M(m'|(x, \xi)|)}$ for some, small enough, $m' > 0$, which completes the proof in the $\{M_p\}$ case. \square

Theorem 3.2. Let $\sum_{j \in \mathbb{N}} a_j \in FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ be given. Than, there exists $a \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$, such that $a \sim \sum_{j \in \mathbb{N}} a_j$.

Proof. Define $\varphi(x) \in \mathcal{D}^{(B_p)}(\mathbb{R}^d)$ and $\psi(\xi) \in \mathcal{D}^{(A_p)}(\mathbb{R}^d)$, in the (M_p) case, resp. $\varphi(x) \in \mathcal{D}^{\{B_p\}}(\mathbb{R}^d)$ and $\psi(\xi) \in \mathcal{D}^{\{A_p\}}(\mathbb{R}^d)$ in the $\{M_p\}$ case, such that $0 \leq \varphi, \psi \leq 1$, $\varphi(x) = 1$ when $\langle x \rangle \leq 2$, $\psi(\xi) = 1$ when $\langle \xi \rangle \leq 2$ and $\varphi(x) = 0$ when $\langle x \rangle \geq 3$, $\psi(\xi) = 0$ when $\langle \xi \rangle \geq 3$. Put $\chi(x, \xi) = \varphi(x)\psi(\xi)$, $\chi_n(x, \xi) = \chi\left(\frac{x}{Rm_n}, \frac{\xi}{Rm_n}\right)$ for $n \in \mathbb{Z}_+$ and $R > 0$ and put $\chi_0(x, \xi) = 0$. It is easily checked that $\chi, \chi_n \in \mathcal{D}^{(M_p)}(\mathbb{R}^{2d})$, resp. $\chi, \chi_n \in \mathcal{D}^{\{M_p\}}(\mathbb{R}^{2d})$.

The (M_p) case. Let $m, B > 0$ are chosen such that $\sum_j a_j \in FS_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; B, h, m)$ for all $h > 0$. For $R \geq 2B$, $a(x, \xi) = \sum_j (1 - \chi_j(x, \xi)) a_j(x, \xi)$ is a well defined $\mathcal{C}^\infty(\mathbb{R}^{2d})$ function. We will prove that for sufficiently large R , $a \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ and $a \sim \sum_j a_j(x, \xi)$ which will complete the proof in the (M_p) case. For $0 < h < 1$, using the fact that $1 - \chi_j(x, \xi) = 0$ for $(x, \xi) \in Q_{Rm_j}$, we have the estimates

$$\begin{aligned}
& \frac{|D_\xi^\alpha D_x^\beta a(x, \xi)| \langle (x, \xi) \rangle^{\rho|\alpha|+\rho|\beta|} e^{-M(m|\xi|)} e^{-M(m|x|)}}{(8h)^{|\alpha|+|\beta|} A_\alpha B_\beta} \\
& \leq \sum_{j \in \mathbb{N}} \sum_{\substack{\gamma \leq \alpha \\ \delta \leq \beta}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} |D_\xi^{\alpha-\gamma} D_x^{\beta-\delta} a_j(x, \xi)| e^{-M(m|\xi|)} e^{-M(m|x|)} \\
& \quad \cdot \frac{|D_\xi^\gamma D_x^\delta (1 - \chi_j(x, \xi))| \langle (x, \xi) \rangle^{\rho|\alpha|+\rho|\beta|}}{(8h)^{|\alpha|+|\beta|} A_\alpha B_\beta} \\
& \leq C_0 \sum_{j \in \mathbb{N}} \sum_{\substack{\gamma \leq \alpha \\ \delta \leq \beta}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \frac{h^{|\alpha|-|\gamma|+|\beta|-|\delta|+2j} A_{\alpha-\gamma} B_{\beta-\delta} A_j B_j}{(8h)^{|\alpha|+|\beta|} A_\alpha B_\beta} \\
& \quad \cdot \langle (x, \xi) \rangle^{\rho|\gamma|+\rho|\delta|-2\rho j} |D_\xi^\gamma D_x^\delta (1 - \chi_j(x, \xi))| \\
& \leq C_0 \sum_{j \in \mathbb{N}} \frac{1}{8^{|\alpha|+|\beta|}} h^{2j} L^{2j} M_j^{2\rho} |1 - \chi_j(x, \xi)| \langle (x, \xi) \rangle^{-2\rho j} \\
& \quad + C_0 \sum_{j \in \mathbb{N}} \frac{1}{8^{|\alpha|+|\beta|}} \sum_{\substack{\gamma \leq \alpha, \delta \leq \beta \\ (\delta, \gamma) \neq (0,0)}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \frac{h^{2j} L^{2j} M_j^{2\rho} |D_\xi^\gamma D_x^\delta (1 - \chi_j(x, \xi))| \langle (x, \xi) \rangle^{\rho|\gamma|+\rho|\delta|-2\rho j}}{h^{|\gamma|+|\delta|} A_\gamma B_\delta} \\
& = S_1 + S_2,
\end{aligned}$$

where S_1 and S_2 are the first and the second sum, correspondingly. To estimate S_1 note that, on the support of $1 - \chi_j$ the inequality $\langle (x, \xi) \rangle \geq Rm_j$ holds. One obtains

$$S_1 \leq C_0 \sum_{j \in \mathbb{N}} \frac{(hL)^{2j} M_j^{2\rho}}{R^{2\rho j} m_j^{2\rho j}} \leq C_0 \sum_{j \in \mathbb{N}} \frac{(hL)^{2j}}{R^{2\rho j}} < \infty,$$

for large enough R (in the second inequality we use the fact that $m_j^j \geq M_j$). For the estimate of S_2 , note that $D_\xi^\gamma D_x^\delta (1 - \chi_j(x, \xi)) = 0$ when $(x, \xi) \in Q_{3Rm_j}^c$, because $(\delta, \gamma) \neq (0, 0)$ and $\chi_j(x, \xi) = 0$ on $Q_{3Rm_j}^c$. So, for $(x, \xi) \in Q_{3Rm_j}$, we have that $\langle (x, \xi) \rangle \leq \langle x \rangle + \langle \xi \rangle \leq 6Rm_j$. Moreover, from the construction of χ , we have that for the chosen h , there exists $C_1 > 0$ such that $|D_\xi^\alpha D_x^\beta \chi(x, \xi)| \leq C_1 h^{|\alpha|+|\beta|} A_\alpha B_\beta$. By using $m_j^j \geq M_j$, one obtains

$$S_2 \leq C_2 \sum_{j \in \mathbb{N}} \frac{1}{8^{|\alpha|+|\beta|}} \sum_{\substack{\gamma \leq \alpha, \delta \leq \beta \\ (\delta, \gamma) \neq (0,0)}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \frac{(hL)^{2j} 6^{\rho|\gamma|+\rho|\delta|} M_j^{2\rho} (Rm_j)^{\rho|\gamma|+\rho|\delta|}}{R^{2\rho j} m_j^{2\rho j} (Rm_j)^{|\gamma|+|\delta|}} \leq C_3 \sum_{j \in \mathbb{N}} \frac{(hL)^{2j}}{R^{2\rho j}},$$

which is convergent for large enough R . Hence, we get that $a \in \Gamma_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; 8h, m)$ for all $0 < h < 1$, from what we obtain $a \in \Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d})$. Now, to prove that $a \sim \sum_{j \in \mathbb{N}} a_j(x, \xi)$.

Note that, for $(x, \xi) \in Q_{3Rm_N}^c$, $a - \sum_{j < N} a_j = \sum_{j \geq N} (1 - \chi_j) a_j$. This easily follows from the

definition of χ_j and the fact that m_n is monotonically increasing.

$$\begin{aligned}
& \frac{\left| D_\xi^\alpha D_x^\beta \sum_{j \geq N} (1 - \chi_j(x, \xi)) a_j(x, \xi) \right| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + 2\rho N} e^{-M(m|\xi|)} e^{-M(m|x|)}}{(8(1+H)h)^{|\alpha| + |\beta| + 2N} A_\alpha B_\beta A_N B_N} \\
& \leq \sum_{j \geq N} \frac{(1 - \chi_j(x, \xi)) \left| D_\xi^\alpha D_x^\beta a_j(x, \xi) \right| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + 2\rho N} e^{-M(m|\xi|)} e^{-M(m|x|)}}{(8(1+H)h)^{|\alpha| + |\beta| + 2N} A_\alpha B_\beta A_N B_N} \\
& \quad + \sum_{j \geq N} \sum_{\substack{\gamma \leq \alpha, \delta \leq \beta \\ (\delta, \gamma) \neq (0, 0)}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \left| D_\xi^{\alpha-\gamma} D_x^{\beta-\delta} a_j(x, \xi) \right| e^{-M(m|\xi|)} e^{-M(m|x|)} \\
& \quad \cdot \frac{\left| D_\xi^\gamma D_x^\delta (1 - \chi_j(x, \xi)) \right| \langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + 2\rho N}}{(8(1+H)h)^{|\alpha| + |\beta| + 2N} A_\alpha B_\beta A_N B_N} \\
& \leq C_0 \sum_{j \geq N} \frac{(1 - \chi_j(x, \xi)) h^{2j-2N} A_j B_j}{(1+H)^{2N} \langle (x, \xi) \rangle^{2\rho j - 2\rho N} A_N B_N} \\
& \quad + C_0 \sum_{j \geq N} \frac{1}{8^{|\alpha| + |\beta|}} \sum_{\substack{\gamma \leq \alpha, \delta \leq \beta \\ (\delta, \gamma) \neq (0, 0)}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \frac{h^{2j-2N} \left| D_\xi^\gamma D_x^\delta (1 - \chi_j(x, \xi)) \right| \langle (x, \xi) \rangle^{\rho|\gamma| + \rho|\delta|} A_j B_j}{(1+H)^{2N} h^{|\gamma| + |\delta|} \langle (x, \xi) \rangle^{2\rho j - 2\rho N} A_\gamma B_\delta A_N B_N} \\
& = S_1 + S_2,
\end{aligned}$$

where S_1 and S_2 are the first and the second sum, correspondingly. To estimate S_1 , observe that on the support of $1 - \chi_j$ the inequality $\langle (x, \xi) \rangle \geq Rm_j$ holds. Using the monotone increasingness of m_n and (M.2) for A_p and B_p , one obtains

$$\begin{aligned}
S_1 & \leq C'_0 \sum_{j \geq N} \frac{h^{2j-2N} H^{2j} A_{j-N} B_{j-N}}{(1+H)^{2N} R^{2\rho j - 2\rho N} m_j^{2\rho j - 2\rho N}} \leq C_4 \sum_{j \geq N} \frac{h^{2j-2N} H^{2j} L^{2j-2N} M_{j-N}^{2\rho}}{(1+H)^{2N} R^{2\rho j - 2\rho N} m_{j-N}^{2\rho j - 2\rho N}} \\
& = C_4 \frac{H^{2N}}{(1+H)^{2N}} \sum_{j=0}^{\infty} \left(\frac{hHL}{R^\rho} \right)^{2j} \leq C_4 \sum_{j=0}^{\infty} \left(\frac{hHL}{R^\rho} \right)^{2j} < \infty,
\end{aligned}$$

uniformly, for $N \in \mathbb{Z}_+$, for large enough R . For S_2 , note that $D_\xi^\gamma D_x^\delta (1 - \chi_j(x, \xi)) = 0$ when $(x, \xi) \in Q_{3Rm_j}^c$, because $(\delta, \gamma) \neq (0, 0)$ and $\chi_j(x, \xi) = 0$ on $Q_{3Rm_j}^c$. Moreover, from the construction of χ , we have that for the chosen h , there exists $C_1 > 0$ such that $|D_\xi^\alpha D_x^\beta \chi(x, \xi)| \leq C_1 h^{|\alpha| + |\beta|} A_\alpha B_\beta$. Now

$$\begin{aligned}
S_2 & \leq C_5 \sum_{j \geq N} \frac{1}{8^{|\alpha| + |\beta|}} \sum_{\substack{\gamma \leq \alpha, \delta \leq \beta \\ (\delta, \gamma) \neq (0, 0)}} \binom{\alpha}{\gamma} \binom{\beta}{\delta} \frac{h^{2j-2N} 6^{|\gamma| + |\delta|} H^{2j} A_{j-N} B_{j-N}}{(1+H)^{2N} R^{2\rho j - 2\rho N} m_j^{2\rho j - 2\rho N}} \\
& \leq C_6 \sum_{j \geq N} \frac{h^{2j-2N} H^{2j} A_{j-N} B_{j-N}}{(1+H)^{2N} R^{2\rho j - 2\rho N} m_j^{2\rho j - 2\rho N}},
\end{aligned}$$

which we already proved that is bounded uniformly for $N \in \mathbb{Z}_+$. Hence, we obtained

$$\sup_{N \in \mathbb{Z}_+} \sup_{\alpha, \beta} \sup_{(x, \xi) \in Q_{3Rm_N}^c} \left| D_\xi^\alpha D_x^\beta \sum_{j \geq N} (1 - \chi_j(x, \xi)) a_j(x, \xi) \right| \cdot \frac{\langle (x, \xi) \rangle^{\rho|\alpha| + \rho|\beta| + 2\rho N} e^{-M(m|\xi|)} e^{-M(m|x|)}}{(8(1+H)h)^{|\alpha| + |\beta| + 2N} A_\alpha B_\beta A_N B_N} < \infty,$$

for arbitrary $h > 0$, i.e. $a \sim \sum_{j \in \mathbb{N}} a_j(x, \xi)$. For the $\{M_p\}$ case, let $h, B > 0$ are such that $a \in FS_{A_p, B_p, \rho}^{M_p, \infty}(\mathbb{R}^{2d}; B, h, m)$ for all $m > 0$. Then, for $R \geq 2B$ we define $a(x, \xi) = \sum_{j \in \mathbb{N}} (1 - \chi_j(x, \xi)) a_j(x, \xi)$ and similarly as above, one proves that, for sufficiently large R , a satisfies the claim in the theorem. \square

Now we will prove theorems for change of quantization and composition of operators. Note that, unlike in [3] and [4], we do not impose additional conditions on A_p and B_p in the composition theorem.

Theorem 3.3. *Let $\tau, \tau_1 \in \mathbb{R}$ and $a \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$. There exists $b \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ and $*$ -regularizing operator T such that $\text{Op}_{\tau_1}(a) = \text{Op}_\tau(b) + T$. Moreover,*

$$b(x, \xi) \sim \sum_{\beta} \frac{1}{\beta!} (\tau_1 - \tau)^{|\beta|} \partial_\xi^\beta D_x^\beta a(x, \xi), \text{ in } FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d}).$$

Proof. Put $p_j(x, \xi) = \sum_{|\beta|=j} \frac{1}{\beta!} (\tau_1 - \tau)^{|\beta|} \partial_\xi^\beta D_x^\beta a(x, \xi)$. One easily verifies that $\sum_j p_j \in FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$. Take the sequence $\chi_j(x, \xi)$, $j \in \mathbb{N}$, constructed in the proof of theorem 3.2, such that $b = \sum_j (1 - \chi_j) p_j$ is an element of $\Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ and $b \sim \sum_j p_j$. By the observations after theorem 2.2, the operators $\text{Op}_{\tau_1}(a)$ and $\text{Op}_\tau(b)$ coincide with the operators A and B corresponding to a and b when we observe $a((1 - \tau_1)x + \tau_1 y, \xi)$ and $b((1 - \tau)x + \tau y, \xi)$ as elements of $\Pi_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{3d})$. It is clear that it is enough to prove that the kernel of $A - B$ is in $\mathcal{S}^*(\mathbb{R}^{2d})$. To prove that, write

$$\begin{aligned} &= \chi_0((1 - \tau)x + \tau y, \xi) a((1 - \tau_1)x + \tau_1 y, \xi) + \sum_{n=0}^{\infty} ((\chi_{n+1} - \chi_n)((1 - \tau)x + \tau y, \xi)) \\ &\quad \cdot \left(a((1 - \tau_1)x + \tau_1 y, \xi) - \sum_{j=0}^n p_j((1 - \tau)x + \tau y, \xi) \right). \end{aligned}$$

By construction $\chi_0 = 0$, so $\chi_0 a = 0$. Note that the above sum is locally finite and it converges in $\mathcal{E}^*(\mathbb{R}^{3d})$. Denote by A_n the operator corresponding to

$$a_n(x, y, \xi) = (\chi_{n+1} - \chi_n)((1 - \tau)x + \tau y, \xi)$$

$$\cdot \left(a((1 - \tau_1)x + \tau_1 y, \xi) - \sum_{j=0}^n p_j((1 - \tau)x + \tau y, \xi) \right)$$

considered as an element of $\Pi_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{3d})$. For $u \in \mathcal{S}^*(\mathbb{R}^d)$, we obtain

$$= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} e^{i(x-y)\xi} \frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) \left(\sum_{n=0}^{\infty} a_n(x, y, \xi) u(y) \right) \right) dy d\xi,$$

in the (M_p) case and the same but with P_{l_p} in place of P_l in the $\{M_p\}$ case. Note that, because of the convergence of the sum in $\mathcal{E}^*(\mathbb{R}^{3d})$, we can interchange the sum with the ultradifferential operators and with $1/P_l(y-x)$ and $1/P_l(\xi)$, resp. with $1/P_{l_p}(y-x)$ and $1/P_{l_p}(\xi)$. For $v \in \mathcal{S}^*(\mathbb{R}^d)$, by the way we define p_j and using the fact about the support of χ_n , with similar technic as in the proof of lemma 2.2, one proves that

$$\sum_{n=0}^{\infty} \int_{\mathbb{R}^{3d}} \left| \frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) (a_n(x, y, \xi) u(y)) \right) v(x) \right| dy d\xi dx < \infty,$$

for sufficiently small l and sufficiently large R (from the definition of χ_n) in the (M_p) case, resp. the same but with P_{l_p} in place of P_l for sufficiently small $(l_p) \in \mathfrak{R}$ and sufficiently large R (from the definition of χ_n) in the $\{M_p\}$ case. Hence, from monotone and dominated convergence it follows that

$$\langle Au - Bu, v \rangle$$

$$\begin{aligned} &= \frac{1}{(2\pi)^d} \sum_{n=0}^{\infty} \int_{\mathbb{R}^{3d}} e^{i(x-y)\xi} \frac{1}{P_l(\xi)} P_l(D_y) \left(\frac{1}{P_l(y-x)} P_l(D_\xi) (a_n(x, y, \xi) u(y)) \right) v(x) dy d\xi dx \\ &= \frac{1}{(2\pi)^d} \sum_{n=0}^{\infty} \int_{\mathbb{R}^{3d}} e^{i(x-y)\xi} a_n(x, y, \xi) u(y) v(x) dy d\xi dx = \sum_{n=0}^{\infty} \langle A_n u, v \rangle \end{aligned}$$

in the (M_p) case, resp. the same but with P_{l_p} in place of P_l in the $\{M_p\}$ case. Hence,

$\sum_{k=0}^n A_k u \rightarrow Au - Bu$, when $n \rightarrow \infty$ in $\mathcal{S}'^*(\mathbb{R}^d)$ for every fixed $u \in \mathcal{S}^*(\mathbb{R}^d)$. But

then, because \mathcal{S}^* is barreled, by the Banach - Steinhaus theorem (see [17], theorem 4.6),

$\sum_{k=0}^n A_k \rightarrow Au - Bu$, when $n \rightarrow \infty$ in the topology of precompact convergence in

$\mathcal{L}(\mathcal{S}^*(\mathbb{R}^d), \mathcal{S}'^*(\mathbb{R}^d))$. \mathcal{S}^* is Montel, hence the convergence holds in $\mathcal{L}_b(\mathcal{S}^*(\mathbb{R}^d), \mathcal{S}'^*(\mathbb{R}^d))$.

If we denote by K and K_n , $n \in \mathbb{N}$, the kernels of the operators $A - B$ and A_n , $n \in \mathbb{N}$ cor-

respondingly, then, by proposition 1.2, it follows that $K = \sum_{n=0}^{\infty} K_n$, where the convergence

is in $\mathcal{S}'^*(\mathbb{R}^{2d})$. Let $r = 1/(8(1 + |\tau| + |\tau_1|))$. Take $\theta \in \mathcal{E}^*(\mathbb{R}^{2d})$ as in lemma 2.5 and put $\tilde{\theta} = 1 - \theta$. θ and $\tilde{\theta}$ are obviously multipliers for \mathcal{S}'^* . By proposition 2.3 and the properties

of θ , $\theta K \in \mathcal{S}^*(\mathbb{R}^{2d})$. It is enough to prove that $\tilde{\theta}K \in \mathcal{S}^*(\mathbb{R}^{2d})$. Note that $\tilde{\theta}K = \sum_n \tilde{\theta}K_n$. Our goal is to prove that $\sum_n \tilde{\theta}K_n \in \mathcal{S}^*$. Observe that

$$K_n(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} (\chi_{n+1} - \chi_n) ((1-\tau)x + \tau y, \xi) \cdot \left(a((1-\tau_1)x + \tau_1 y, \xi) - \sum_{j=0}^n p_j((1-\tau)x + \tau y, \xi) \right) d\xi,$$

for all $n \in \mathbb{N}$. Put $\begin{cases} x' = (1-\tau)x + \tau y, \\ y' = x - y, \end{cases}$ from what we obtain $\begin{cases} x = x' + \tau y', \\ y = x' - (1-\tau)y'. \end{cases}$ Hence $a((1-\tau_1)x + \tau_1 y, \xi) = a(x' + (\tau - \tau_1)y', \xi)$. If we Taylor expand the right hand side in $y' = 0$, we get

$$a((1-\tau_1)x + \tau_1 y, \xi) = \sum_{|\beta| \leq n} \frac{1}{\beta!} (\tau - \tau_1)^{|\beta|} \partial_x^\beta a(x', \xi) (x - y)^\beta + W_{n+1}(x, y, \xi),$$

where W_{n+1} is the reminder of the expansion:

$$W_{n+1}(x, y, \xi) = (n+1) \sum_{|\beta|=n+1} \frac{1}{\beta!} (x - y)^\beta (\tau - \tau_1)^{|\beta|} \int_0^1 (1-t)^n \partial_x^\beta a(x' + t(\tau - \tau_1)y', \xi) dt.$$

If we insert the above expression for a in the expression for K_n we obtain

$$\begin{aligned} K_n(x, y) &= \frac{1}{(2\pi)^d} \sum_{|\beta| \leq n} \frac{1}{\beta!} (\tau - \tau_1)^{|\beta|} \int_{\mathbb{R}^d} e^{i(x-y)\xi} (-D_\xi)^\beta ((\chi_{n+1} - \chi_n)(x', \xi) \partial_x^\beta a(x', \xi)) d\xi \\ &\quad + \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} (\chi_{n+1} - \chi_n)(x', \xi) W_{n+1}(x, y, \xi) d\xi \\ &\quad - \frac{1}{(2\pi)^d} \sum_{j=0}^n \int_{\mathbb{R}^d} e^{i(x-y)\xi} (\chi_{n+1} - \chi_n)(x', \xi) p_j((1-\tau)x + \tau y, \xi) d\xi \\ &= S_{1,n}(x, y) + S_{2,n}(x, y) - S_{3,n}(x, y). \end{aligned}$$

Our goal is to prove that each of the sums $\sum_n \tilde{\theta}(S_{1,n} - S_{3,n})$ and $\sum_n \tilde{\theta}S_{2,n}$, is \mathcal{S}^* function. Because of the way we defined p_j , one obtains

$$S_{1,n}(x, y) - S_{3,n}(x, y)$$

$$= \frac{1}{(2\pi)^d} \sum_{0 \neq |\beta| \leq n} \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta!} (\tau_1 - \tau)^{|\beta|} \int_{\mathbb{R}^d} e^{i(x-y)\xi} (D_\xi^\delta (\chi_{n+1} - \chi_n))(x', \xi) D_\xi^{\beta-\delta} \partial_x^\beta a(x', \xi) d\xi.$$

Put

$$\tilde{S}_{\beta,n}(x, y) = \frac{1}{(2\pi)^d} \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta!} (\tau_1 - \tau)^{|\beta|}$$

$$\cdot \int_{\mathbb{R}^d} e^{i(x-y)\xi} (D_\xi^\delta (\chi_{n+1} - \chi_n)) (x', \xi) D_\xi^{\beta-\delta} \partial_x^\beta a(x', \xi) d\xi.$$

Obviously $\tilde{S}_{\beta,n} \in \mathcal{E}^* (\mathbb{R}^{2d}) \cap \mathcal{S}'^* (\mathbb{R}^{2d})$. Let $w \in \mathcal{S}^* (\mathbb{R}^{2d})$. Note that

$$\begin{aligned} \langle \tilde{S}_{\beta,n}, w \rangle &= \frac{1}{(2\pi)^d} \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta!} (\tau_1 - \tau)^{|\beta|} \int_{\mathbb{R}^{3d}} \frac{1}{P_l(\xi)} e^{i(x-y)\xi} \\ &\quad \cdot P_l(D_y) \left((D_\xi^\delta (\chi_{n+1} - \chi_n)) (x', \xi) D_\xi^{\beta-\delta} \partial_x^\beta a(x', \xi) w(x, y) \right) d\xi dx dy, \end{aligned}$$

in the (M_p) case, where $l > 0$ will be chosen later, resp. the same but with P_{l_p} in place of P_l in the $\{M_p\}$ case, where $(l_p) \in \mathfrak{R}$ will be chosen later. We will consider first the (M_p) case. Then there exists $m > 0$ such that $a \in \Gamma_{A_p, B_p, \rho}^{(M_p), \infty} (\mathbb{R}^{2d}; m)$. Chose l such that $|P_l(\xi)| \geq c' e^{4M(m|\xi|)}$ (cf. proposition 1.1). On the other hand $P_l(\xi) = \sum_\alpha c_\alpha \xi^\alpha$ and there exist $C_0 > 0$ and $L_0 > 0$ such that $|c_\alpha| \leq C_0 L_0^{|\alpha|} / M_\alpha$. Note that, when $(\chi_{n+1} - \chi_n) (x', \xi) \neq 0$, $\langle (x', \xi) \rangle \geq Rm_n$. Using this, one easily obtains that

$$\begin{aligned} &\sum_{n=1}^{\infty} \sum_{0 \neq |\beta| \leq n} \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{(|\tau_1| + |\tau|)^{|\beta|}}{\beta!} \\ &\quad \cdot \int_{\mathbb{R}^{3d}} \left| \frac{e^{i(x-y)\xi}}{P_l(\xi)} P_l(D_y) \left((D_\xi^\delta (\chi_{n+1} - \chi_n)) (x', \xi) D_\xi^{\beta-\delta} \partial_x^\beta a(x', \xi) w(x, y) \right) \right| d\xi dx dy < \infty, \end{aligned}$$

for sufficiently large R (from the definition of χ_n). In the $\{M_p\}$ case, by lemma 2.1 there exists $(k_p) \in \mathfrak{R}$ such that the estimate in that lemma holds (we can regard $a((1-\tau)x + \tau y, \xi)$ as an element of $\Pi_{A_p, B_p, \rho}^{\{M_p\}, \infty} (\mathbb{R}^{3d}; h)$). Take $(l_p) \in \mathfrak{R}$ such that $|P_{l_p}(\xi)| \geq c' e^{4N_{k_p}(|\xi|)}$. One obtains the same estimate as above but with P_{l_p} in place of P_l , for sufficiently large R (from the definition of χ_n). From this we obtain that $\sum_{n=1}^{\infty} (S_{1,n} - S_{3,n}) = \sum_{n=1}^{\infty} \sum_{0 \neq |\beta| \leq n} \tilde{S}_{\beta,n}$ converges

in $\mathcal{S}'^* (\mathbb{R}^{2d})$. Denote its limit by $\tilde{S}(x, y)$. Moreover, from the above, we can change the order of summation and integration. The local finiteness of $\sum_n (\chi_{n+1} - \chi_n)$ implies

$$\sum_{n \geq |\beta|} D_\xi^\delta (\chi_{n+1}(x', \xi) - \chi_n(x', \xi)) = D_\xi^\delta (1 - \chi_{|\beta|}(x', \xi)) = -D_\xi^\delta \chi_{|\beta|}(x', \xi),$$

where the last equality follows from the fact that $\delta \neq 0$. In the (M_p) case, we obtain

$$\begin{aligned} &\sum_{n=1}^{\infty} \sum_{0 \neq |\beta| \leq n} \langle \tilde{S}_{\beta,n}, w \rangle \\ &= -\frac{1}{(2\pi)^d} \sum_{|\beta|=1}^{\infty} \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta!} (\tau_1 - \tau)^{|\beta|} \\ &\quad \cdot \int_{\mathbb{R}^{3d}} \frac{1}{P_l(\xi)} e^{i(x-y)\xi} P_l(D_y) \left(D_\xi^\delta \chi_{|\beta|}(x', \xi) D_\xi^{\beta-\delta} \partial_x^\beta a(x', \xi) w(x, y) \right) d\xi dx dy \end{aligned}$$

$$= -\frac{1}{(2\pi)^d} \sum_{|\beta|=1}^{\infty} \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta!} (\tau_1 - \tau)^{|\beta|} \int_{\mathbb{R}^{2d}} I_{\beta,\delta}(x, y) w(x, y) dx dy,$$

where we put $I_{\beta,\delta}(x, y) = \int_{\mathbb{R}^d} e^{i(x-y)\xi} D_{\xi}^{\delta} \chi_{|\beta|}(x', \xi) D_{\xi}^{\beta-\delta} \partial_x^{\beta} a(x', \xi) d\xi$. Similarly, in the $\{M_p\}$ case we obtain the same equality. Hence $-\frac{1}{(2\pi)^d} \sum_{|\beta|=1}^{\infty} \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta!} (\tau_1 - \tau)^{|\beta|} I_{\beta,\delta}(x, y)$ converges to $\tilde{S}(x, y)$ in $\mathcal{S}^*(\mathbb{R}^{2d})$. Now we will prove that $\tilde{\theta}\tilde{S}$ is \mathcal{S}^* function. Denote

$$T_n = \{(x, \xi) \in \mathbb{R}^{2d} \mid |x| \leq 3Rm_n \text{ and } |\xi| \leq 3Rm_n\} \quad (13)$$

and put $T_{\xi,n}$ to be the projection of T_n on \mathbb{R}_{ξ}^d . By construction $\text{supp } \chi_{|\beta|} \subseteq T_{|\beta|}$. So, for the derivatives of $I_{\beta,\delta}(x, y)$ when $(x, y) \in \mathbb{R}^{2d} \setminus \Omega_r \supseteq \text{supp } \tilde{\theta}$, we have

$$\begin{aligned} & \left| D_x^{\beta'} D_y^{\gamma'} I_{\beta,\delta}(x, y) \right| \\ & \leq \sum_{\substack{\alpha \leq \beta' \\ \nu \leq \gamma'}} \sum_{\substack{\alpha' + \alpha'' = \alpha \\ \nu' + \nu'' = \nu}} \binom{\gamma'}{\nu} \binom{\beta'}{\alpha} \binom{\alpha}{\alpha'} \binom{\nu}{\nu'} (1 + |\tau|)^{|\alpha'| + |\nu'|} (1 + |\tau|)^{|\beta'| - |\alpha| + |\gamma'| - |\nu|} \\ & \quad \cdot \int_{T_{\xi, |\beta|}} |\xi|^{|\alpha''| + |\nu''|} \left| D_{\xi}^{\delta} D_x^{\alpha' + \nu'} \chi_{|\beta|}(x', \xi) \right| \left| D_{\xi}^{\beta - \delta} D_x^{\beta + \gamma' - \nu + \beta' - \alpha} a(x', \xi) \right| d\xi \\ & \leq C_1 \sum_{\substack{\alpha \leq \beta' \\ \nu \leq \gamma'}} \sum_{\substack{\alpha' + \alpha'' = \alpha \\ \nu' + \nu'' = \nu}} \binom{\gamma'}{\nu} \binom{\beta'}{\alpha} \binom{\alpha}{\alpha'} \binom{\nu}{\nu'} (1 + |\tau|)^{|\beta'| + |\gamma'| - |\alpha''| - |\nu''|} \\ & \quad \cdot \int_{T_{\xi, |\beta|}} |\xi|^{|\alpha''| + |\nu''|} \frac{h_1^{|\delta| + |\alpha'| + |\nu'|} A_{\delta} B_{\alpha' + \nu'}}{(Rm_{|\beta|})^{|\delta| + |\alpha'| + |\nu'|}} \\ & \quad \cdot \frac{h_1^{2\beta - \delta + \beta' + \gamma' - \alpha - \nu} A_{\beta - \delta} B_{\beta + \beta' + \gamma' - \alpha - \nu} e^{M(m|\xi|)} e^{M(m|x'|)}}{\langle (x', \xi) \rangle^{\rho|2\beta - \delta + \beta' + \gamma' - \alpha - \nu|}} d\xi. \end{aligned}$$

Because $\delta \neq 0$, $D_{\xi}^{\delta} D_x^{\alpha' + \nu'} \chi_{|\beta|}(x', \xi) = 0$ when $\chi_{|\beta|}(x', \xi) = 1$, hence when $|x'| \leq Rm_{|\beta|}$ and $|\xi| \leq Rm_{|\beta|}$. So, when $D_{\xi}^{\delta} D_x^{\alpha' + \nu'} \chi_{|\beta|}(x', \xi) \neq 0$ we have $\langle (x', \xi) \rangle \geq Rm_{|\beta|}$. We obtain

$$R^{|\delta| + |\alpha'| + |\nu'|} m_{|\beta|}^{|\delta| + |\alpha'| + |\nu'|} \langle (x', \xi) \rangle^{\rho|2\beta - \delta + \beta' + \gamma' - \alpha - \nu|} \geq (Rm_{|\beta|})^{\rho|2\beta + \beta' + \gamma' - \alpha'' - \nu''|}.$$

By assumption, there exists $c, L \geq 1$ such that $A_p \leq cL^p M_p^{\rho}$ and $B_p \leq cL^p M_p^{\rho}$. Hence

$$\begin{aligned} & \frac{A_{\delta} B_{\alpha' + \nu'} A_{\beta - \delta} B_{\beta + \beta' + \gamma' - \alpha - \nu}}{(Rm_{|\beta|})^{\rho|2\beta + \beta' + \gamma' - \alpha'' - \nu''|}} \\ & \leq \frac{A_{\beta} B_{\beta + \beta' + \gamma' - \alpha'' - \nu''}}{(Rm_{|\beta|})^{\rho|2\beta + \beta' + \gamma' - \alpha'' - \nu''|}} \leq \frac{c^2 L^{2\beta + \beta' + \gamma' - \alpha'' - \nu''} M_{2\beta + \beta' + \gamma' - \alpha'' - \nu''}^{\rho}}{(Rm_{|\beta|})^{2\rho|\beta|} (Rm_{|\beta|})^{\rho|\beta' + \gamma' - \alpha'' - \nu''|}} \end{aligned}$$

$$\leq \frac{C'''(LH^2)^{|2\beta+\beta'+\gamma'-\alpha''-\nu''|} M_{\beta}^{2\rho} M_{\beta'+\gamma'}}{R^{2\rho|\beta|} m_{|\beta|}^{2\rho|\beta|} (RM_1)^{\rho|\beta'+\gamma'-\alpha''-\nu''|} M_{\alpha''+\nu''}} \leq \frac{C'''(LH^2)^{|2\beta+\beta'+\gamma'-\alpha''-\nu''|} M_{\beta'+\gamma'}}{R^{2\rho|\beta|} (RM_1)^{\rho|\beta'+\gamma'-\alpha''-\nu''|} M_{\alpha''+\nu''}},$$

where, in the last inequality, we used that $m_n^n \geq M_n$. Also, note that when $(x, y) \in \mathbb{R}^{2d} \setminus \Omega_r$ and $\chi_{|\beta|}((1-\tau)x + \tau y, \xi) \neq 0$, we have the following inequalities

$$\begin{aligned} |x'| &= |(1-\tau)x + \tau y| \leq 3Rm_{|\beta|}, \\ |x|^2 + |y|^2 &\leq 2|x|^2 + |x-y|^2 + 2|x||x-y| \leq 2|x|^2 + r^2\langle x \rangle^2 + 2r|x|\langle x \rangle \\ &\leq (2+r)^2\langle x \rangle^2, \end{aligned}$$

$$1 + |(1-\tau)x + \tau y|^2 \geq 1 + |x|^2 + |\tau|^2|x-y|^2 - 2|\tau||x||x-y| \geq \langle x \rangle^2 - \frac{1}{4}\langle x \rangle^2 \geq \frac{1}{4}\langle x \rangle^2,$$

(remember, $r = 1/(8(1+|\tau|+|\tau_1|))$). Put $s = 2+r$ for shorter notation. Combining these inequalities we get $|(x, y)| \leq 2s\langle x' \rangle \leq 8sRm_{|\beta|}$. Using this and proposition 3.6 of [9], for arbitrary $m' > 0$, we obtain

$$e^{M(m|x'|)} \leq e^{M(3mRm_{|\beta|})} e^{M(8sm'Rm_{|\beta|})} e^{-M(m'|(x,y)|)} \leq c_0 e^{M(8s(m+m')HRm_{|\beta|})} e^{-M(m'|(x,y)|)},$$

when $(x, y) \in \mathbb{R}^{2d} \setminus \Omega_r$ and $\chi_{|\beta|}((1-\tau)x + \tau y, \xi) \neq 0$. Using these inequalities in the estimate for $D_x^{\beta'} D_y^{\gamma'} I_{\beta,\delta}(x, y)$, for $(x, y) \in \mathbb{R}^{2d} \setminus \Omega_r$, we get

$$\begin{aligned} &\left| D_x^{\beta'} D_y^{\gamma'} I_{\beta,\delta}(x, y) \right| \\ &\leq C_3 \sum_{\substack{\alpha \leq \beta' \\ \nu \leq \gamma'}} \sum_{\substack{\alpha' + \alpha'' = \alpha \\ \nu' + \nu'' = \nu}} \binom{\gamma'}{\nu} \binom{\beta'}{\alpha} \binom{\alpha}{\alpha'} \binom{\nu}{\nu'} (1 + |\tau|)^{|\beta'|+|\gamma'|-|\alpha''|-\nu''|} \\ &\quad \cdot \int_{T_{\xi,|\beta|}} |\xi|^{|\alpha''|+|\nu''|} \frac{(LH^2)^{|2\beta+\beta'+\gamma'-\alpha''-\nu''|} h_1^{|\delta|+|\alpha'|+|\nu'|} h^{2\beta-\delta+\beta'+\gamma'-\alpha-\nu} M_{\beta'+\gamma'}}{R^{2\rho|\beta|} (RM_1)^{\rho|\beta'+\gamma'-\alpha''-\nu''|} M_{\alpha''+\nu''}} \\ &\quad \cdot e^{M(m|\xi|)} e^{M(8s(m+m')HRm_{|\beta|})} e^{-M(m'|(x,y)|)} d\xi \\ &\leq C_4 \frac{M_{\beta'+\gamma'}}{e^{M(m'|(x,y)|)}} \sum_{\substack{\alpha \leq \beta' \\ \nu \leq \gamma'}} \sum_{\substack{\alpha' + \alpha'' = \alpha \\ \nu' + \nu'' = \nu}} \binom{\gamma'}{\nu} \binom{\beta'}{\alpha} \binom{\alpha}{\alpha'} \binom{\nu}{\nu'} \\ &\quad \cdot \int_{T_{\xi,|\beta|}} \frac{(LH^2)^{2|\beta|} h_1^{|\delta|+|\alpha'|+|\nu'|} h^{2\beta-\delta+\beta'+\gamma'-\alpha-\nu} e^{2M((m+m')R|\xi|)} e^{M(8s(m+m')HRm_{|\beta|})}}{(m'R)^{|\alpha''|+|\nu''|} R^{2\rho|\beta|}} d\xi, \end{aligned}$$

where, in the last inequality, we used that

$$\frac{(1 + |\tau|)^{|\beta'|+|\gamma'|-|\alpha''|-\nu''|} (LH^2)^{|\beta'|+|\gamma'|-|\alpha''|-\nu''|}}{(RM_1)^{\rho|\beta'+\gamma'-\alpha''-\nu''|}} \leq 1,$$

for large enough R . Moreover, on $T_{\xi,|\beta|}$, by proposition 3.6 of [9], we have $2M((m+m')R|\xi|) \leq M(3(m+m')HR^2m_{|\beta|}) + \ln c_0$. Lemma 2.3 implies

$$\begin{aligned} &e^{M(3(m+m')HR^2m_{|\beta|})} e^{M(8s(m+m')HRm_{|\beta|})} \\ &\leq c_0^2 H^{2(3c_0(m+m')HR^2+2)|\beta|} H^{2(8c_0s(m+m')HR+2)|\beta|} \leq c_0^2 H^{4(8c_0s(m+m')HR^2+2)|\beta|}. \end{aligned}$$

Similarly as in the proof for proposition 2.3, we have $|T_{\xi,|\beta|}| \leq C_5 R^d H^{d|\beta|}$, for some $C_5 > 0$. For the (M_p) case, m is fixed. It is clear that, without losing generality, we can assume that $m \geq 1$. Choose R such that $R \geq 4$ and $R^{2\rho} \geq 2(1 + |\tau| + |\tau_1|)L^2 H^{d+4}$. For arbitrary but fixed $m' > 0$, choose h such that $hH^{4(8c_0s(m+m')HR^2+2)} \leq 1$ and $2h \leq 1/(4m')$. Moreover, choose h_1 such that $h_1 \leq h$. Then we obtain

$$\begin{aligned} \left| D_x^{\beta'} D_y^{\gamma'} I_{\beta,\delta}(x, y) \right| &\leq C_6 R^d \frac{M_{\beta'+\gamma'} h^{|\beta|}}{e^{M(m'|(x,y)|)}} \left(\frac{L^2 H^{d+4}}{R^{2\rho}} \right)^{|\beta|} \left(2h + \frac{1}{m'R} \right)^{|\beta'|+|\gamma'|} \\ &\leq C_6 R^d \frac{M_{\beta'+\gamma'} h^{|\beta|}}{e^{M(m'|(x,y)|)}} \cdot \frac{1}{(2(1 + |\tau| + |\tau_1|))^{|\beta|}} \cdot \frac{1}{(2m')^{|\beta'|+|\gamma'|}}, \end{aligned}$$

when $(x, y) \in \mathbb{R}^{2d} \setminus \Omega_r$. Note that the choice of R (and hence of χ_n , $n \in \mathbb{N}$) depends only on A_p , B_p , M_p , τ , τ_1 and a , but not on m' . By the definition of $\tilde{\theta}$ it follows that there exists $C' > 0$ such that

$$\left| D_x^{\beta'} D_y^{\gamma'} \left(\tilde{\theta}(x, y) I_{\beta,\delta}(x, y) \right) \right| \leq C' R^d \frac{M_{\beta'+\gamma'} h^{|\beta|}}{e^{M(m'|(x,y)|)}} \cdot \frac{1}{(2(1 + |\tau| + |\tau_1|))^{|\beta|}} \cdot \frac{1}{m'^{|\beta'|+|\gamma'|}},$$

for all $(x, y) \in \mathbb{R}^{2d}$ and $\beta', \gamma' \in \mathbb{N}^d$. Hence

$$\sum_{|\beta|=1}^{\infty} \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta!} (|\tau_1| + |\tau|)^{|\beta|} \left| D_x^{\beta'} D_y^{\gamma'} \left(\tilde{\theta}(x, y) I_{\beta,\delta}(x, y) \right) \right| \leq C \frac{M_{\beta'+\gamma'}}{m'^{|\beta'|+|\gamma'|} e^{M(m'|(x,y)|)}},$$

for all $(x, y) \in \mathbb{R}^{2d}$ and $\beta', \gamma' \in \mathbb{N}^d$. From the arbitrariness of m' it follows that $\tilde{\theta}\tilde{S} \in \mathcal{S}^{(M_p)}$. Now we consider the $\{M_p\}$ case. Then h and h_1 are fixed. Choose R such that $R^{2\rho} \geq 2(1 + |\tau| + |\tau_1|)(h + h_1)hL^2 H^{d+16}$ and then choose m and m' such that $8c_0s(m+m')HR^2 \leq 1$. Then $H^{4(8c_0s(m+m')HR^2+2)|\beta|} \leq H^{12|\beta|}$. Then we have

$$\begin{aligned} \left| D_x^{\beta'} D_y^{\gamma'} I_{\beta,\delta}(x, y) \right| &\leq C_6 R^d \frac{M_{\beta'+\gamma'} (hL^2 H^{d+16})^{|\beta|} h_1^{|\delta|} h^{|\beta-\delta|}}{e^{M(m'|(x,y)|)} R^{2\rho|\beta|}} \left(h + h_1 + \frac{1}{m'R} \right)^{|\beta'|+|\gamma'|} \\ &\leq C_6 R^d \frac{M_{\beta'+\gamma'}}{e^{M(m'|(x,y)|)} (2(1 + |\tau| + |\tau_1|))^{|\beta|}} \left(h + h_1 + \frac{1}{m'R} \right)^{|\beta'|+|\gamma'|}, \end{aligned}$$

when $(x, y) \in \mathbb{R}^{2d} \setminus \Omega_r$. By the definition of $\tilde{\theta}$ it follows that there exist $C' > 0$ and $\tilde{h} > 0$ such that

$$\left| D_x^{\beta'} D_y^{\gamma'} \left(\tilde{\theta}(x, y) I_{\beta,\delta}(x, y) \right) \right| \leq C' R^d \frac{M_{\beta'+\gamma'} \tilde{h}^{|\beta'|+|\gamma'|}}{e^{M(m'|(x,y)|)} (2(1 + |\tau| + |\tau_1|))^{|\beta|}},$$

for all $(x, y) \in \mathbb{R}^{2d}$ and $\beta', \gamma' \in \mathbb{N}^d$. Hence

$$\sum_{|\beta|=1}^{\infty} \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta!} (|\tau_1| + |\tau|)^{|\beta|} \left| D_x^{\beta'} D_y^{\gamma'} \left(\tilde{\theta}(x, y) I_{\beta,\delta}(x, y) \right) \right| \leq C \frac{M_{\beta'+\gamma'} \tilde{h}^{|\beta'|+|\gamma'|}}{e^{M(m'|(x,y)|)}},$$

for all $(x, y) \in \mathbb{R}^{2d}$ and $\beta', \gamma' \in \mathbb{N}^d$, from what we obtain $\tilde{\theta}\tilde{S} \in \mathcal{S}^{\{M_p\}}$.

It remains to prove that $\sum_{n=0}^{\infty} \tilde{\theta}(x, y) S_{2,n}(x, y) \in \mathcal{S}^*$. Note that

$$S_{2,n}(x, y) = \frac{n+1}{(2\pi)^d} \sum_{|\beta|=n+1} \sum_{\delta \leq \beta} \binom{\beta}{\delta} \frac{(-1)^{|\beta|}}{\beta!} (\tau - \tau_1)^{|\beta|} \int_{\mathbb{R}^d} e^{i(x-y)\xi} D_{\xi}^{\delta} (\chi_{n+1} - \chi_n)(x', \xi) \\ \cdot \int_0^1 (1-t)^n D_{\xi}^{\beta-\delta} \partial_x^{\beta} a(x' + t(\tau - \tau_1)y', \xi) dt d\xi.$$

For brevity in notation, put

$$\tilde{I}_{\beta, \delta, n}(x, y) = \int_{\mathbb{R}^d} e^{i(x-y)\xi} D_{\xi}^{\delta} (\chi_{n+1} - \chi_n)(x', \xi) \int_0^1 (1-t)^n D_{\xi}^{\beta-\delta} \partial_x^{\beta} a(x' + t(\tau - \tau_1)y', \xi) dt d\xi.$$

We will estimate $\left| D_x^{\beta'} D_y^{\gamma'} \tilde{I}_{\beta, \delta, n}(x, y) \right|$ when $(x, y) \in \mathbb{R}^{2d} \setminus \Omega_r \supseteq \text{supp } \tilde{\theta}$.

$$\left| D_x^{\beta'} D_y^{\gamma'} \tilde{I}_{\beta, \delta, n}(x, y) \right| \\ \leq C_1 \sum_{\substack{\alpha \leq \beta' \\ \nu \leq \gamma'}} \sum_{\substack{\alpha' + \alpha'' = \alpha \\ \nu' + \nu'' = \nu}} \binom{\beta'}{\alpha} \binom{\gamma'}{\nu} \binom{\alpha}{\alpha'} \binom{\nu}{\nu'} (2(1 + |\tau| + |\tau_1|))^{| \beta' | - | \alpha'' | + | \gamma' | - | \nu'' |} \\ \cdot \int_{T_{\xi, n+1}} |\xi|^{| \alpha'' | + | \nu'' |} \frac{h_1^{|\delta| + | \alpha' | + | \nu' |} A_{\delta} B_{\alpha' + \nu'}}{(Rm_n)^{| \alpha' | + | \nu' | + | \delta |}} \\ \cdot \int_0^1 (1-t)^n \frac{h^{2|\beta| - |\delta| + |\beta'| - |\alpha| + |\gamma'| - |\nu|} A_{\beta - \delta} B_{\beta + \beta' - \alpha + \gamma' - \nu} e^{M(m|\xi|)} e^{M(m|x' + t(\tau - \tau_1)y'|)}}{\langle (x' + t(\tau - \tau_1)y', \xi) \rangle^{\rho(2|\beta| - |\delta| + |\beta'| - |\alpha| + |\gamma'| - |\nu|)}} dt d\xi.$$

Above, we already proved that on $\mathbb{R}^{2d} \setminus \Omega_r$, $\langle x \rangle \leq 2\langle x' \rangle$. Using this, by similar technic as there, one easily proves that $\langle (x' + t(\tau - \tau_1)y', \xi) \rangle \geq Rm_n$ when $(x, y) \in \mathbb{R}^{2d} \setminus \Omega_r$ and $\chi_{n+1}(x', \xi) - \chi(x', \xi) \neq 0$. Also, for such x, y and ξ we have $|x' + t(\tau - \tau_1)y'| \leq |x'| + (|\tau| + |\tau_1|)|y'| \leq \langle x' \rangle + 2r(|\tau| + |\tau_1|)\langle x' \rangle \leq 8Rm_{n+1}$ and $|\xi| \leq 3Rm_{n+1}$. Now, the proof continues analogously as above and one obtains $\sum_n \tilde{\theta} S_{2,n} \in \mathcal{S}^*$. We already pointed out that from this it follows that $K \in \mathcal{S}^*$. \square

Theorem 3.4. *Let $\tau \in \mathbb{R}$ and $a \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$. The transposed operator, ${}^t\text{Op}_{\tau}(a)$, is still a pseudo-differential operator and it is equal to $\text{Op}_{1-\tau}(a(x, -\xi))$. Moreover, there exist $b \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ and $*$ -regularizing operator T such that ${}^t\text{Op}_{\tau}(a) = \text{Op}_{\tau}(b) + T$ and*

$$b(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} (1 - 2\tau)^{|\alpha|} (-\partial_{\xi})^{\alpha} D_x^{\alpha} a(x, -\xi) \text{ in } FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d}).$$

Proof. In the observation after theorem 2.2 we proved that ${}^t\text{Op}_{\tau}(a(x, \xi)) = \text{Op}_{1-\tau}(a(x, -\xi))$. The rest follows from theorem 3.3. \square

Theorem 3.5. *Let $a, b \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$. There exist $f \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ and *-regularizing operator T such that $a(x, D)b(x, D) = f(x, D) + T$ and f has the asymptotic expansion*

$$f(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_{\xi}^{\alpha} a(x, \xi) D_x^{\alpha} b(x, \xi) \text{ in } FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d}). \quad (14)$$

Proof. By the above theorem ${}^t b(x, D) = b_1(x, D) + T'$ where T' is *-regularizing operator and $b_1 \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ with asymptotic expansion

$$b_1(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} (-\partial_{\xi})^{\alpha} D_x^{\alpha} b(x, -\xi). \quad (15)$$

Again, by the above theorem, ${}^t b_1(x, D) = \text{Op}_1(b_1(x, -\xi))$ and $b(x, D) = {}^t \text{Op}_1(b(x, -\xi)) = {}^t ({}^t b(x, D))$. Put $b_2(x, \xi) = b_1(x, -\xi)$. Then we have

$$b(x, D) = {}^t ({}^t b(x, D)) = {}^t b_1(x, D) + {}^t T' = \text{Op}_1(b_2) + {}^t T'.$$

We have $a(x, D)b(x, D) = a(x, D)\text{Op}_1(b_2) + T_1$, where we put $T_1 = a(x, D){}^t T'$, which is *-regularizing. Because $\mathcal{F}(\text{Op}_1(b_2)u)(\xi) = \int_{\mathbb{R}^d} e^{-iy\xi} b_2(y, \xi) u(y) dy$ and $\text{Op}_1(b_2)u \in \mathcal{S}^*$,

$$a(x, D)\text{Op}_1(b_2)u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} a(x, \xi) b_2(y, \xi) u(y) dy d\xi$$

and this is well defined as iterated integral by theorem 2.1. $\tilde{a}(x, y, \xi) = a(x, \xi) b_2(y, \xi)$ is an element of $\Pi_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{3d})$. To prove that one only has to use the inequalities $2\langle(x, \xi)\rangle\langle x - y \rangle \geq \langle(x, y, \xi)\rangle$ and $2\langle(y, \xi)\rangle\langle x - y \rangle \geq \langle(x, y, \xi)\rangle$ in the estimates for the derivatives of \tilde{a} . The operator \tilde{A} corresponding to this \tilde{a} is the same as $a(x, D)\text{Op}_1(b_2)$. Let

$$p_j(x, \xi) = \sum_{|\beta|=j} \frac{1}{\beta!} \partial_{\xi}^{\beta} (a(x, \xi) D_x^{\beta} b_2(x, \xi)).$$

Obviously $\sum_j p_j \in FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$. Let $\chi_j(x, \xi)$, $j \in \mathbb{N}$, be the sequence constructed in the proof of theorem 3.2, such that $f = \sum_j (1 - \chi_j) p_j$ is an element of $\Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ and $f \sim \sum_j p_j$. By the observations after theorem 2.2, the operator $f(x, D)$ coincide with the operator F corresponding to f when we observe $f(x, \xi)$ as elements of $\Pi_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{3d})$. We will prove that the kernel of $\tilde{A} - F$ is in $\mathcal{S}^*(\mathbb{R}^{2d})$, i.e. $\tilde{A} - F$ is *-regularizing. Similarly as in the proof of theorem 3.3, $\tilde{a}(x, y, \xi) - f(x, \xi) = \sum_{n=0}^{\infty} \tilde{a}_n(x, y, \xi)$ where we put

$$\tilde{a}_n(x, y, \xi) = (\chi_{n+1}(x, \xi) - \chi_n(x, \xi)) \left(\tilde{a}(x, y, \xi) - \sum_{j=0}^n p_j(x, \xi) \right),$$

which is obviously an element of $\Pi_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{3d})$. Denote by \tilde{A}_n its corresponding operator. Similarly as in the proof of theorem 3.3, we have $K(x, y) = \sum_n K_n(x, y)$, where K is the kernel of $\tilde{A} - F$, K_n is the kernel of \tilde{A}_n and the convergence holds in \mathcal{S}'^* . Observe that

$$K_n(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} (\chi_{n+1} - \chi_n)(x, \xi) \left(a(x, \xi) b_2(y, \xi) - \sum_{j=0}^n p_j(x, \xi) \right) d\xi,$$

for all $n \in \mathbb{N}$. Let $r = 1/8$. Take $\theta \in \mathcal{E}^*(\mathbb{R}^{2d})$ as in lemma 2.5 and put $\tilde{\theta} = 1 - \theta$. θ and $\tilde{\theta}$ are obviously multipliers for \mathcal{S}'^* . By proposition 2.3 and the properties of θ , $\theta K \in \mathcal{S}^*(\mathbb{R}^{2d})$. It is enough to prove that $\tilde{\theta} K \in \mathcal{S}^*(\mathbb{R}^{2d})$. Note that $\tilde{\theta} K = \sum_n \tilde{\theta} K_n$. Our goal is to prove that $\sum_n \tilde{\theta} K_n \in \mathcal{S}^*$. Taylor expand $b_2(y, \xi)$ in the first variable to obtain

$$b_2(y, \xi) = \sum_{|\beta| \leq n} \frac{1}{\beta!} (y - x)^\beta \partial_x^\beta b_2(x, \xi) + W_{n+1}(x, y, \xi),$$

where W_{n+1} is the remainder of the expansion:

$$W_{n+1}(x, y, \xi) = (n+1) \sum_{|\beta|=n+1} \frac{1}{\beta!} (y - x)^\beta \int_0^1 (1-t)^n \partial_x^\beta b_2(x + t(y-x), \xi) dt.$$

If we insert this in the expression for K_n , keeping in mind the definition of p_j , we have $K_n(x, y) = S_{1,n}(x, y) + S_{2,n}(x, y)$ where we put

$$\begin{aligned} S_{1,n}(x, y) &= \frac{1}{(2\pi)^d} \sum_{0 \neq |\beta| \leq n} \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta!} \\ &\quad \cdot \int_{\mathbb{R}^d} e^{i(x-y)\xi} D_\xi^\delta (\chi_{n+1} - \chi_n)(x, \xi) D_\xi^{\beta-\delta} (a(x, \xi) \partial_x^\beta b_2(x, \xi)) d\xi \\ S_{2,n}(x, y) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(x-y)\xi} (\chi_{n+1} - \chi_n)(x, \xi) a(x, \xi) W_{n+1}(x, y, \xi) d\xi. \end{aligned}$$

Our goal is to prove that $\sum_n \tilde{\theta} S_{1,n}$ and $\sum_n \tilde{\theta} S_{2,n}$ are \mathcal{S}^* functions. Similarly as in the proof of theorem 3.3, $\sum_n S_{1,n}$ converges in \mathcal{S}'^* to \tilde{S} and $\tilde{S} = -\frac{1}{(2\pi)^d} \sum_{|\beta|=1}^\infty \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta!} I_{\beta, \delta}$, where the convergence is in \mathcal{S}'^* , where we put

$$I_{\beta, \delta}(x, y) = \int_{\mathbb{R}^d} e^{i(x-y)\xi} D_\xi^\delta \chi_{|\beta|}(x, \xi) D_\xi^{\beta-\delta} (a(x, \xi) \partial_x^\beta b_2(x, \xi)) d\xi.$$

To prove that $-\frac{1}{(2\pi)^d} \sum_{|\beta|=1}^\infty \sum_{0 \neq \delta \leq \beta} \binom{\beta}{\delta} \frac{1}{\beta!} \tilde{\theta} I_{\beta, \delta}$ is in \mathcal{S}^* we have to estimate the derivatives of $I_{\beta, \delta}$ when $(x, y) \in \mathbb{R}^{2d} \setminus \Omega_r \supseteq \text{supp } \tilde{\theta}$. Note that, we can choose m such that $a, b_2 \in$

$\Gamma_{A_p, B_p, \rho}^{(M_p), \infty}(\mathbb{R}^{2d}; m)$ in the (M_p) case, resp. we can choose h such that $a, b_2 \in \Gamma_{A_p, B_p, \rho}^{\{M_p\}, \infty}(\mathbb{R}^{2d}; h)$ in the $\{M_p\}$ case. Let T_n be as in (13) and put $T_{\xi, n}$ to be the projection of T_n on \mathbb{R}_{ξ}^d . By the way we constructed χ_n , it follows that $\text{supp } \chi_{|\beta|} \subseteq T_{|\beta|}$.

$$\begin{aligned}
& \left| D_x^{\beta'} D_y^{\gamma'} I_{\beta, \delta}(x, y) \right| \\
& \leq \sum_{\kappa \leq \beta - \delta} \sum_{\alpha \leq \beta'} \sum_{\alpha' + \alpha'' = \alpha} \sum_{\alpha''' \leq \beta' - \alpha} \binom{\beta - \delta}{\kappa} \binom{\beta'}{\alpha} \binom{\alpha}{\alpha'} \binom{\beta' - \alpha}{\alpha'''} \\
& \quad \cdot \int_{T_{\xi, |\beta|}} |\xi|^{| \alpha'' | + | \gamma' |} \left| D_{\xi}^{\delta} D_x^{\alpha'} \chi_{|\beta|}(x, \xi) \right| \left| D_{\xi}^{\beta - \delta - \kappa} D_x^{\beta' - \alpha - \alpha'''} a(x, \xi) D_{\xi}^{\kappa} D_x^{\beta + \alpha'''} b_2(x, \xi) \right| d\xi \\
& \leq C_1 \sum_{\kappa \leq \beta - \delta} \sum_{\alpha \leq \beta'} \sum_{\alpha' + \alpha'' = \alpha} \binom{\beta - \delta}{\kappa} \binom{\beta'}{\alpha} \binom{\alpha}{\alpha'} 2^{|\beta'| - |\alpha|} \\
& \quad \cdot \int_{T_{\xi, |\beta|}} |\xi|^{| \alpha'' | + | \gamma' |} \frac{h_1^{|\delta| + |\alpha'|} A_{\delta} B_{\alpha'} h^{|\beta - \delta + \beta' - \alpha|} A_{\beta - \delta} B_{\beta + \beta' - \alpha} e^{2M(m|\xi|)} e^{2M(m|x|)}}{(Rm_{|\beta|})^{|\delta| + |\alpha'|} \langle (x, \xi) \rangle^{\rho|\beta - \delta + \beta' - \alpha|}} d\xi.
\end{aligned}$$

Because $\delta \neq 0$, $D_{\xi}^{\delta} D_x^{\alpha'} \chi_{|\beta|}(x, \xi) = 0$ when $\chi_{|\beta|}(x, \xi) = 1$, hence when $|x| \leq Rm_{|\beta|}$ and $|\xi| \leq Rm_{|\beta|}$. So, when $D_{\xi}^{\delta} D_x^{\alpha'} \chi_{|\beta|}(x, \xi) \neq 0$ we have $\langle (x, \xi) \rangle \geq Rm_{|\beta|}$. Now the proof continues analogously as for theorem 3.3.

Next we will prove that $\sum_n \tilde{\theta}(x, y) S_{2, n}(x, y) \in \mathcal{S}^*$. Note that

$$\begin{aligned}
S_{2, n}(x, y) &= \frac{n+1}{(2\pi)^d} \sum_{|\beta|=n+1} \sum_{\delta \leq \beta} \sum_{\kappa \leq \beta - \delta} \binom{\beta}{\delta} \binom{\beta - \delta}{\kappa} \frac{1}{\beta!} \int_{\mathbb{R}^d} e^{i(x-y)\xi} D_{\xi}^{\delta} (\chi_{n+1} - \chi_n)(x, \xi) \\
& \quad \cdot D_{\xi}^{\kappa} a(x, \xi) \int_0^1 (1-t)^n D_{\xi}^{\beta - \delta - \kappa} \partial_x^{\beta} b_2(x + t(y-x), \xi) dt d\xi.
\end{aligned}$$

For brevity in notation, put

$$\begin{aligned}
\tilde{I}_{\beta, \delta, n}(x, y) &= \sum_{\kappa \leq \beta - \delta} \binom{\beta - \delta}{\kappa} \int_{\mathbb{R}^d} e^{i(x-y)\xi} D_{\xi}^{\delta} (\chi_{n+1} - \chi_n)(x, \xi) D_{\xi}^{\kappa} a(x, \xi) \\
& \quad \cdot \int_0^1 (1-t)^n D_{\xi}^{\beta - \delta - \kappa} \partial_x^{\beta} b_2(x + t(y-x), \xi) dt d\xi.
\end{aligned}$$

We will estimate $\left| D_x^{\beta'} D_y^{\gamma'} \tilde{I}_{\beta, \delta, n}(x, y) \right|$ when $(x, y) \in \mathbb{R}^{2d} \setminus \Omega_r \supseteq \text{supp } \tilde{\theta}$.

$$\begin{aligned}
& \left| D_x^{\beta'} D_y^{\gamma'} \tilde{I}_{\beta, \delta, n}(x, y) \right| \\
& \leq C_1 \sum_{\substack{\alpha \leq \beta' \\ \nu \leq \gamma'}} \sum_{\alpha' + \alpha'' = \alpha} \sum_{\kappa \leq \beta - \delta} \sum_{\alpha''' \leq \beta' - \alpha} \binom{\beta'}{\alpha} \binom{\gamma'}{\nu} \binom{\alpha}{\alpha'} \binom{\beta - \delta}{\kappa} \binom{\beta' - \alpha}{\alpha'''} \\
& \quad \cdot \int_{T_{\xi, n+1}} |\xi|^{| \alpha'' | + | \nu |} \frac{h_1^{|\delta| + |\alpha'|} A_{\delta} B_{\alpha'}}{(Rm_n)^{|\alpha'| + |\delta|}}
\end{aligned}$$

$$\cdot \int_0^1 (1-t)^n \frac{h^{2|\beta|+|\delta|+|\beta'|+|\alpha|+|\gamma'|+|\nu|} A_{\beta-\delta} B_{\beta+\beta'-\alpha+\gamma'-\nu} e^{2M(m|\xi|)} e^{2M(m(|x|+|y|))}}{\langle(x, \xi)\rangle^{\rho(|\beta'|+|\alpha|+|\alpha''|+|\kappa|)} \langle(x+t(y-x), \xi)\rangle^{\rho(2|\beta|+|\delta|+|\kappa|+|\alpha''|+|\gamma'|+|\nu|)}} dt d\xi.$$

When $(\chi_{n+1} - \chi_n)(x, \xi) \neq 0$ and $(x, y) \in \mathbb{R}^{2d} \setminus \Omega_r$, the inequalities $\langle(x, \xi)\rangle \geq Rm_m$ and $\langle(x+t(y-x), \xi)\rangle \geq Rm_m$ hold. Also $|x| + |y| \leq 2|x| + |x-y| \leq s\langle x \rangle \leq 4sRm_{n+1}$, where we put $s = 2 + r$. Hence

$$\begin{aligned} & \left| D_x^{\beta'} D_y^{\gamma'} \tilde{I}_{\beta, \delta, n}(x, y) \right| \\ & \leq \frac{C_2}{n+1} \sum_{\substack{\alpha \leq \beta' \\ \nu \leq \gamma'}} \sum_{\alpha' + \alpha'' = \alpha} \binom{\beta'}{\alpha} \binom{\gamma'}{\nu} \binom{\alpha}{\alpha'} 2^{|\beta|+|\delta|+|\beta'|+|\alpha|} \int_{T_{\xi, n+1}} |\xi|^{|\alpha''|+|\nu|} d\xi \\ & \quad \cdot \frac{h_1^{|\delta|+|\alpha'|} h^{2|\beta|+|\delta|+|\beta'|+|\alpha|+|\gamma'|+|\nu|} A_{\beta} B_{\beta+\beta'-\alpha''+\gamma'-\nu} e^{M(3mHRm_{n+1})} e^{M(4smHRm_{n+1})}}{(Rm_n)^{\rho(2|\beta|+|\beta'|+|\alpha''|+|\gamma'|+|\nu|)}}. \end{aligned}$$

The proof continues in analogous fashion as for theorem 3.3 and one obtains that $\sum_n \tilde{\theta} S_{2,n} \in \mathcal{S}^*$. Hence, we proved that $a(x, D)b(x, D) = a(x, D)\text{Op}_1(b_2) + T_1 = f(x, D) + T_2$, where T_2 is *-regularizing operator. It remains to prove (14). Obviously, it is enough to prove that

$$\begin{aligned} & \sum_{\beta} \frac{1}{\beta!} \partial_{\xi}^{\beta} (a(x, \xi) D_x^{\beta} b_2(x, \xi)) \sim \sum_{\beta} \frac{1}{\beta!} \partial_{\xi}^{\beta} a(x, \xi) D_x^{\beta} b(x, \xi). \text{ For } N \in \mathbb{Z}_+ \text{ we have} \\ & \sum_{j=0}^{N-1} \sum_{|\beta|=j} \frac{1}{\beta!} \partial_{\xi}^{\beta} (a \cdot D_x^{\beta} b_2) \\ & = \sum_{j=0}^{N-1} \sum_{|\alpha+\gamma|=j} \frac{1}{\alpha! \gamma!} \partial_{\xi}^{\gamma} a \cdot \left(\partial_{\xi}^{\alpha} D_x^{\alpha+\gamma} b_2 - \sum_{s=0}^{N-j-1} \sum_{|\delta|=s} \frac{(-1)^{|\delta|}}{\delta!} \partial_{\xi}^{\alpha+\delta} D_x^{\alpha+\gamma+\delta} b \right) \\ & \quad + \sum_{j=0}^{N-1} \sum_{s=0}^{N-j-1} \sum_{|\alpha+\gamma|=j} \sum_{|\delta|=s} \frac{(-1)^{|\delta|}}{\alpha! \gamma! \delta!} \partial_{\xi}^{\gamma} a \cdot \partial_{\xi}^{\alpha+\delta} D_x^{\alpha+\gamma+\delta} b. \end{aligned}$$

Note that

$$\begin{aligned} & \sum_{j=0}^{N-1} \sum_{s=0}^{N-j-1} \sum_{|\alpha+\gamma|=j} \sum_{|\delta|=s} \frac{(-1)^{|\delta|}}{\alpha! \gamma! \delta!} \partial_{\xi}^{\gamma} a \cdot \partial_{\xi}^{\alpha+\delta} D_x^{\alpha+\gamma+\delta} b \\ & = \sum_{j=0}^{N-1} \sum_{s=0}^{N-j-1} \sum_{k=0}^j \sum_{\substack{|\alpha|=k \\ |\gamma|=j-k}} \sum_{|\delta|=s} \frac{(-1)^{|\delta|}}{\alpha! \gamma! \delta!} \partial_{\xi}^{\gamma} a \cdot \partial_{\xi}^{\alpha+\delta} D_x^{\alpha+\gamma+\delta} b \\ & = \sum_{s=0}^{N-1} \sum_{j=0}^{N-s-1} \sum_{k=0}^j \sum_{\substack{|\alpha|=k \\ |\gamma|=j-k}} \sum_{|\delta|=s} \frac{(-1)^{|\delta|}}{\alpha! \gamma! \delta!} \partial_{\xi}^{\gamma} a \cdot \partial_{\xi}^{\alpha+\delta} D_x^{\alpha+\gamma+\delta} b \\ & = \sum_{s=0}^{N-1} \sum_{k=0}^{N-s-1} \sum_{j=k}^{N-s-1} \sum_{\substack{|\alpha|=k \\ |\gamma|=j-k}} \sum_{|\delta|=s} \frac{(-1)^{|\delta|}}{\alpha! \gamma! \delta!} \partial_{\xi}^{\gamma} a \cdot \partial_{\xi}^{\alpha+\delta} D_x^{\alpha+\gamma+\delta} b \end{aligned}$$

$$\begin{aligned}
&= \sum_{t=0}^{N-1} \sum_{s+k=t} \sum_{j=k}^{N-s-1} \sum_{|\gamma|=j-k} \sum_{\substack{|\alpha|=k \\ |\delta|=s}} \frac{(-1)^{|\delta|}}{\alpha! \gamma! \delta!} \partial_\xi^\gamma a \cdot \partial_\xi^{\alpha+\delta} D_x^{\alpha+\gamma+\delta} b \\
&= \sum_{t=0}^{N-1} \sum_{s+k=t} \sum_{j=k}^{N-s-1} \sum_{|\gamma|=j-k} \sum_{|\beta|=t} \sum_{\alpha+\delta=\beta} \frac{(-1)^{|\delta|}}{\alpha! \gamma! \delta!} \partial_\xi^\gamma a \cdot \partial_\xi^\beta D_x^{\beta+\gamma} b \\
&= \sum_{j=0}^{N-1} \sum_{|\gamma|=j} \frac{1}{\gamma!} \partial_\xi^\gamma a \cdot D_x^\gamma b.
\end{aligned}$$

Hence, we have to estimate the derivatives of

$$\sum_{j=0}^{N-1} \sum_{|\alpha+\gamma|=j} \frac{1}{\alpha! \gamma!} \partial_\xi^\gamma a \cdot \partial_\xi^\alpha D_x^{\alpha+\gamma} \left(b_2 - \sum_{s=0}^{N-j-1} \sum_{|\delta|=s} \frac{(-1)^{|\delta|}}{\delta!} \partial_\xi^\delta D_x^\delta b \right).$$

By construction $b(x, \xi) \sim \sum_{j=0}^{\infty} \sum_{|\delta|=j} \frac{(-1)^{|\delta|}}{\delta!} \partial_\xi^\delta D_x^\delta b(x, \xi)$. So, for $(x, \xi) \in Q_{Bm_N}^c$, we have

$$\begin{aligned}
&\left| D_\xi^{\alpha'} D_x^{\beta'} \sum_{j=0}^{N-1} \sum_{|\alpha+\gamma|=j} \frac{1}{\alpha! \gamma!} \partial_\xi^\gamma a(x, \xi) \partial_\xi^\alpha D_x^{\alpha+\gamma} \left(b_2(x, \xi) - \sum_{s=0}^{N-j-1} \sum_{|\delta|=s} \frac{(-1)^{|\delta|}}{\delta!} \partial_\xi^\delta D_x^\delta b(x, \xi) \right) \right| \\
&\leq C_1 \sum_{j=0}^{N-1} \sum_{|\alpha+\gamma|=j} \sum_{\substack{\alpha'' \leq \alpha' \\ \beta'' \leq \beta'}} \binom{\alpha'}{\alpha''} \binom{\beta'}{\beta''} \frac{h^{|\alpha'|+|\beta'|+2N} A_{|\alpha'|+j} B_{|\beta'|+j} A_{N-j} B_{N-j} e^{2M(m|\xi|)} e^{2M(m|x|)}}{\alpha! \gamma! \langle (x, \xi) \rangle^{\rho(|\alpha'|+|\beta'|+2N)}} \\
&\leq C \frac{(4Hh)^{|\alpha'|+|\beta'|+2N} A_{\alpha'} B_{\beta'} A_N B_N e^{M(mH|\xi|)} e^{M(mH|x|)}}{\langle (x, \xi) \rangle^{\rho(|\alpha'|+|\beta'|+2N)}},
\end{aligned}$$

which gives the desired asymptotic expansion. \square

For the next corollary we need the following technical lemma.

Lemma 3.2. *Let $a, b \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ are such that $a \sim \sum_j a_j$ and $b \sim \sum_j b_j$. Then*

$$ab \sim \sum_{j=0}^{\infty} \sum_{s+k=j} a_s b_k \text{ and}$$

$$\partial_\xi^\alpha a(x, \xi) \partial_x^\alpha b(x, \xi) \sim \underbrace{0 + \dots + 0}_{|\alpha|} + \sum_{j=|\alpha|}^{\infty} \sum_{s+k+|\alpha|=j} \partial_\xi^\alpha a_s(x, \xi) \partial_x^\alpha b_k(x, \xi) \quad (16)$$

in $FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$, for each $\alpha \in \mathbb{N}^d$. Moreover, there exist $B > 0$ and $m > 0$ such that, for every $h > 0$, there exists $C > 0$; resp. there exist $B > 0$ and $h > 0$ such that, for every $m > 0$, there exists $C > 0$; such that

$$\sup_{\alpha} \sup_{N > |\alpha|} \sup_{\gamma, \delta} \sup_{(x, \xi) \in Q_{Bm_N}^c} \left| D_\xi^\gamma D_x^\delta \left(\partial_\xi^\alpha a(x, \xi) \partial_x^\alpha b(x, \xi) - \sum_{j=|\alpha|}^{N-1} \sum_{s+k+|\alpha|=j} \partial_\xi^\alpha a_s(x, \xi) \partial_x^\alpha b_k(x, \xi) \right) \right|$$

$$\frac{\langle (x, \xi) \rangle^{\rho|\gamma|+\rho|\delta|+2N\rho} e^{-M(m|\xi|)} e^{-M(m|x|)}}{h^{|\gamma|+|\delta|+2N} A_\gamma B_\delta A_N B_N} \leq C.$$

Proof. By the conditions in the lemma, there exist $B > 0$ and $m > 0$ such that, for every $h > 0$, there exists $\tilde{C} > 0$; resp. there exist $B > 0$ and $h > 0$ such that, for every $m > 0$, there exists $\tilde{C} > 0$; such that

$$\begin{aligned} \sup_{j \in \mathbb{N}} \sup_{\gamma, \delta} \sup_{(x, \xi) \in Q_{Bm_j}^c} \frac{|D_\xi^\gamma D_x^\delta a_j(x, \xi)| \langle (x, \xi) \rangle^{\rho|\gamma|+\rho|\delta|+2j\rho} e^{-M(m|\xi|)} e^{-M(m|x|)}}{h^{|\gamma|+|\delta|+2j} A_\gamma B_\delta A_j B_j} &\leq \tilde{C}, \\ \sup_{N \in \mathbb{Z}_+} \sup_{\gamma, \delta} \sup_{(x, \xi) \in Q_{Bm_N}^c} \frac{|D_\xi^\gamma D_x^\delta (a(x, \xi) - \sum_{j < N} a_j(x, \xi))| \langle (x, \xi) \rangle^{\rho|\gamma|+\rho|\delta|+2N\rho}}{h^{|\gamma|+|\delta|+2N} A_\gamma B_\delta A_N B_N} &\cdot e^{-M(m|\xi|)} e^{-M(m|x|)} \leq \tilde{C} \end{aligned}$$

and the same estimate for $D_\xi^\gamma D_x^\delta b_j$ and $D_\xi^\gamma D_x^\delta (b - \sum_{j < N} b_j)$. One easily checks that

$$\underbrace{0 + \dots + 0}_{|\alpha|} + \sum_{j=|\alpha|}^{\infty} \sum_{s+k+|\alpha|=j} \partial_\xi^\alpha a_s \partial_x^\alpha b_k \in FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d}), \text{ for each fixed } \alpha \in \mathbb{N}^d. \text{ For } N > |\alpha|$$

and $(x, \xi) \in Q_{Bm_N}^c$, observe that

$$\begin{aligned} \partial_\xi^\alpha a \cdot \partial_x^\alpha b &= \partial_\xi^\alpha a \cdot \left(\partial_x^\alpha b - \sum_{k=0}^{N-|\alpha|-1} \partial_x^\alpha b_k \right) + \sum_{k=0}^{N-|\alpha|-1} \left(\partial_\xi^\alpha a - \sum_{s=0}^{N-|\alpha|-k-1} \partial_\xi^\alpha a_s \right) \cdot \partial_x^\alpha b_k \\ &\quad + \sum_{j=|\alpha|}^{N-1} \sum_{s+k=j-|\alpha|} \partial_\xi^\alpha a_s \partial_x^\alpha b_k. \end{aligned}$$

Using this, one verifies that

$$\begin{aligned} &\left| D_\xi^\gamma D_x^\delta \sum_{k=0}^{N-|\alpha|-1} \left(\partial_\xi^\alpha a(x, \xi) - \sum_{s=0}^{N-|\alpha|-k-1} \partial_\xi^\alpha a_s(x, \xi) \right) \cdot \partial_x^\alpha b_k(x, \xi) \right| \\ &\leq c_0^4 \tilde{C}^2 \frac{(4hH)^{|\gamma|+|\delta|+2N} A_\gamma B_\delta A_N B_N e^{M(mH|\xi|)} e^{M(mH|x|)}}{\langle (x, \xi) \rangle^{\rho|\gamma|+\rho|\delta|+2N\rho}}, \end{aligned}$$

for all $(x, \xi) \in Q_{Bm_N}^c$, $\gamma, \delta \in \mathbb{N}^d$ and the estimates are uniform for α and N , $N > |\alpha|$.

Analogously, one obtains similar estimates for the derivatives of $\partial_\xi^\alpha a \cdot \left(\partial_x^\alpha b - \sum_{k=0}^{N-|\alpha|-1} \partial_x^\alpha b_k \right)$.

Now we can estimate the derivatives of $\partial_\xi^\alpha a \cdot \partial_x^\alpha b - \sum_{j=|\alpha|}^{N-1} \sum_{s+k=j-|\alpha|} \partial_\xi^\alpha a_s \partial_x^\alpha b_k$ and obtain the inequality in the lemma. Moreover, for fixed $\alpha \in \mathbb{N}^d$, to obtain (16) it only remains to

consider the case when $N \leq |\alpha|$ (we already consider the case when $N > |\alpha|$ above).

But then $\sum_{j=|\alpha|}^{N-1} \sum_{s+k=j-|\alpha|} \partial_\xi^\alpha a_s \partial_x^\alpha b_k$ is empty and we only have to estimate the derivatives of $\partial_\xi^\alpha a \cdot \partial_x^\alpha b$ which is easy and we omit it ($a, b \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ and α is fixed). \square

Corollary 3.1. *Let $a, b \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ with asymptotic expansions $a \sim \sum_j a_j$ and $b \sim \sum_j b_j$. Then there exists $c \in \Gamma_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$ and *-regularizing operator T such that $a(x, D)b(x, D) = c(x, D) + T$ and c has the following asymptotic expansion*

$$c(x, \xi) \sim \sum_{j=0}^{\infty} \sum_{s+k+l=j} \sum_{|\alpha|=l} \frac{1}{\alpha!} \partial_\xi^\alpha a_s(x, \xi) D_x^\alpha b_k(x, \xi).$$

Proof. It is easy to check that the above formal sum is an element of $FS_{A_p, B_p, \rho}^{*, \infty}(\mathbb{R}^{2d})$. By theorem 3.5, we only have to prove that

$$\sum_{j=0}^{\infty} \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \xi) D_x^\alpha b(x, \xi) \sim \sum_{j=0}^{\infty} \sum_{s+k+l=j} \sum_{|\alpha|=l} \frac{1}{\alpha!} \partial_\xi^\alpha a_s(x, \xi) D_x^\alpha b_k(x, \xi).$$

For $N \in \mathbb{Z}_+$ and $(x, \xi) \in Q_{Bm_N}^c$, we have

$$\begin{aligned} & \sum_{j=0}^{N-1} \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \xi) D_x^\alpha b(x, \xi) - \sum_{j=0}^{N-1} \sum_{s+k+l=j} \sum_{|\alpha|=l} \frac{1}{\alpha!} \partial_\xi^\alpha a_s(x, \xi) D_x^\alpha b_k(x, \xi) \\ &= \sum_{j=0}^{N-1} \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \xi) D_x^\alpha b(x, \xi) - \sum_{j=0}^{N-1} \sum_{l=0}^j \sum_{s+k=j-l} \sum_{|\alpha|=l} \frac{1}{\alpha!} \partial_\xi^\alpha a_s(x, \xi) D_x^\alpha b_k(x, \xi) \\ &= \sum_{j=0}^{N-1} \sum_{|\alpha|=j} \frac{1}{\alpha!} \partial_\xi^\alpha a(x, \xi) D_x^\alpha b(x, \xi) - \sum_{l=0}^{N-1} \sum_{j=l}^{N-1} \sum_{s+k=j-l} \sum_{|\alpha|=l} \frac{1}{\alpha!} \partial_\xi^\alpha a_s(x, \xi) D_x^\alpha b_k(x, \xi) \\ &= \sum_{j=0}^{N-1} \sum_{|\alpha|=j} \frac{1}{\alpha!} \left(\partial_\xi^\alpha a(x, \xi) D_x^\alpha b(x, \xi) - \sum_{l=j}^{N-1} \sum_{s+k=l-j} \partial_\xi^\alpha a_s(x, \xi) D_x^\alpha b_k(x, \xi) \right). \end{aligned}$$

By lemma 3.2, the derivatives of $\partial_\xi^\alpha a(x, \xi) D_x^\alpha b(x, \xi) - \sum_{l=j}^{N-1} \sum_{s+k=l-j} \partial_\xi^\alpha a_s(x, \xi) D_x^\alpha b_k(x, \xi)$ can be uniformly estimated, as in the lemma, for all α , N and $(x, \xi) \in Q_{Bm_N}^c$, such that $|\alpha| < N$, from what the desired equivalence follows. \square

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